Unstable minimal surfaces of annulus type in manifolds

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Abstract

Unstable minimal surfaces are the unstable stationary points of the Dirichlet integral. In order to obtain unstable solutions, the method of the gradient flow together with the minimax-principle is generally used, an application of which was presented in [St2] for minimal surfaces in Euclidean space. We extend this theory to obtain unstable minimal surfaces in Riemannian manifolds. In particular, we consider minimal surfaces of annulus type.

1 Introduction

For given curves $\Gamma_l \subset N, l = 1, ..., m$ and $\Gamma := \Gamma_1 \cup \cdots \cup \Gamma_m$, where (N, h) is a Riemannian manifold of dimension $n \geq 2$ with metric $(h_{\alpha\beta})$, we denote the generalized Plateau Problem by $\mathcal{P}(\Gamma)$. This deals with minimal surfaces bounded by Γ, in other words parametrizations X defined on $\Sigma \subset \mathbb{R}^2$ with $\partial\Sigma = \Gamma$, satisfying the following constraints:

(1) $\tau_h(X) = 0$,

(2) $|X_u|_h^2 - |X_v|_h^2 = \langle X_u, X_v \rangle_h = 0$,

(3) $X|_{\partial\Sigma}$ is weakly monotone and onto Γ,

where $\tau_h := \Delta X^\alpha - \Gamma^\alpha_{\beta\gamma} \nabla X^\beta X^\gamma = 0$ is the harmonic equation on (N, h) seen as the Euler-Lagrange equation of the energy functional.

A regular minimal surface is called unstable if its surface area is not a minimum among neighbouring surfaces with the same boundary.

Extending the Ljusternik-Schnirelmann theory on convex sets in Banach spaces, a variational approach to unstable minimal surfaces of disc or annulus type in $\mathbb{R}^n$ was proposed in 1983 ([St1], see also [St2] [St3]). For the minimal surfaces of higher topological structure in $\mathbb{R}^n$, it was studied in [JS].

Recently in [Ho], the existence of unstable minimal surfaces of higher topological structure with one boundary in a nonpositively curved Riemannian manifold was studied by applying

[1]This paper is based on my thesis [Ki1] supervised by Professor Michael Grüter

the method in [St2]. In particular, the first part of that paper considers the Jacobi field extension operator as the derivative of the harmonic extension.

In this article, we study unstable minimal surfaces of annulus type in manifolds. The Euclidean case was tackled already in [St3], and our aim is to generalize this result to manifolds satisfying appropriate conditions. Namely, we will consider two boundary curves Γ_1, Γ_2 in a Riemannian manifold (N, h) such that one of the following holds.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p, r)$, where $B(p, r)$ lies within the normal range of all its points. We assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N, h).

(C2) N is compact with nonpositive sectional curvature.

These conditions are related to the existence and uniqueness of the harmonic extension for a given boundary parametrization.

First, we construct suitable spaces of functions, the boundary parametrizations, distinguishing the cases (C1) and (C2). We introduce a convex set which serves as a tangent space for the given boundary parametrization. Then we consider the following functional:

$$\mathcal{E}(x) := \frac{1}{2} \int |d\mathcal{F}(x)|_h^2,$$

where $\mathcal{F}(x)$ denotes the harmonic extension of annulus type or of two-disc type with boundary parametrization x. We next discuss the differentiability of $\mathcal{E}$, in particular for the case in which the topology of the surfaces changes (from an annulus to two discs). Defining critical points of $\mathcal{E}$, will show the equivalence between the harmonic extensions (in N) of critical points of $\mathcal{E}$ and minimal surfaces in N. The $H^{2,2}$-regularity of the harmonic extension of a critical point of $\mathcal{E}$ (see the appendix or [Ki2]) plays an important role in the argument.

In section 4, we prove the Palais-Smale condition for $\mathcal{E}$. In particular, we investigate carefully the behaviour of boundary mappings which are fixed at only one point. In order to deform level sets of $\mathcal{E}$, we also construct a suitable vector field and its corresponding flow.

Roughly speaking, Lemma 4.3 shows that the energy of some annulus-type harmonic extensions is greater than that of two-disc type harmonic extensions by a uniformly positive constant. Although this result refers to Riemannian manifolds, it turns out to be more restrictive than that of Euclidean spaces, which holds uniformly on any bounded set of boundary parametrizations. This somewhat weaker result is anyhow enough for the present purposes.

Following the arguments set out in [St1], we can prove the main theorem of this paper. This states that if there exists a minimal surface (of annulus type) whose energy is a strict relative minimum in $\mathcal{S}(\Gamma_1, \Gamma_2)$ (suitably defined for each case (C1) and (C2)), the existence of an unstable minimal surface of annulus type is ensured under certain assumptions related to the solutions of $\mathcal{P}(\Gamma_i)$. We eventually apply this result to the three-dimensional sphere S^3 and the three-dimensional hyperbolic space H^3, whose curvatures are 1 and -1, respectively.

2 Preliminaries

2.1 Some definitions

Let (N, h) be a connected, oriented, complete Riemannian manifold of dimension $n \geq 2$, embedded isometrically and properly into some $\mathbb{R}^k$ as a closed submanifold by means of the map η ([Gro]). Moreover, $d\omega$ and d_0 denote the area elements in $\Omega \subset \mathbb{R}^2$ and in $\partial\Omega$ respectively.

Indicating

$$B := \{w \in \mathbb{R}^2 \mid |w| < 1\}$$

we define

$$H^{1,2} \cap C^0(B, N) := \{f \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | f(B) \subset N\}$$

with norm $\|f\|_{1,2;0} := \|df\|_{L^2} + \|f\|_{C^0}$. Now set

$$T_f H^{1,2} \cap C^0(B, N) \cong \{V \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | V(\cdot) \in T_{f(\cdot)}N\} =: H^{1,2} \cap C^0(B, f^*TN),$$

with norm

$$\|V\| := \Big(\int_B |\nabla^f V|_h^2 d\omega\Big)^{\frac{1}{2}} + \|V\|_{C^0} \cong \Big(\int_B |dV|^2_{\mathbb{R}^k} d\omega\Big)^{\frac{1}{2}} + \|V\|_{C^0}. \tag{1}$$

Let Γ be a Jordan curve in N diffeomorphic to $S^1 := \partial B$. Then N can be equipped with another metric $\tilde{h}$ such that Γ is a geodesic in $(N, \tilde{h})$. We observe that $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_{\tilde{h}}\big)$ and $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_h\big)$ coincide as sets.

Using the exponential map in $(N, \tilde{h})$, we let

$$H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) \quad := \quad \{u \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k) | u(\partial B) = \Gamma\},$$

where the norm is given by $\|u\|_{\frac{1}{2},2;0} := \|d\mathcal{H}(u)\|_{L^2} + \|u\|_{C^0}$, and $\mathcal{H}(u)$ is the harmonic extension in $\mathbb{R}^k$ with $\mathcal{H}(u)|_{\partial B}(\cdot) = u(\cdot)$. In addition

$$\begin{aligned} T_u H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) \quad &:= \quad \{\xi \in H^{\frac{1}{2},2} \cap C^0(\partial B, u^*TN) | \xi(z) \in T_{u(z)}\Gamma, \text{ for all } z \in \partial B\} \\ &= \quad H^{\frac{1}{2},2} \cap C^0(\partial B, u^*T\Gamma). \end{aligned}$$

Finally, the energy of $f \in H^{1,2}(\Omega, N)$ is denoted by

$$E(f) := \frac{1}{2} \int_\Omega |df|_h^2 dw.$$

2.2 The setting

Let Γ_1, Γ_2 be two Jordan curves of class C^3 in N with diffeomorphisms $\gamma^i : \partial B \to \Gamma_i, i = 1, 2$, and $\text{dist}(\Gamma_1, \Gamma_2) > 0$. For $\rho \in (0, 1)$ let

$$A_\rho := \{w \in B \mid \rho < |w| < 1\}$$

have boundary $C_1 := \partial B$ and $C_\rho := \partial B_\rho =: C_2$ (ρ fixed), and indicate

$$\mathfrak{X}^i_{\text{mon}} := \{x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \,|\, x^i \text{ is weakly monotone and onto } \Gamma_i \text{ with degree } 1\}.$$

I) We first consider the following condition for $(N, h)(\supset \Gamma_1, \Gamma_2)$.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p, r)$, where $B(p, r)$ lies within the normal range of all its points. We assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N, h).

Throughout the paper, $B(p, r)$ denotes a geodesic ball with center $p \in N$ as in (C1).

We can easily observe the following property (see [Ki2]).

Remark 2.1. *If $\Gamma_1, \Gamma_2 \subset N$ satisfy (C1), then for each $x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ and $\rho \in (0, 1)$ there exist $g_\rho \in H^{1,2} \cap C^0(\overline{A_\rho}, B(p, r))$ and $g^i \in H^{1,2} \cap C^0(\overline{B}, B(p, r))$ with $g_\rho|_{C_1} = x^1$, $g_\rho|_{C_\rho}(\cdot) = x^2(\frac{\cdot}{\rho})$ and $g^i|_{\partial B} = x^i$, $i = 1, 2$.*

From the results in [HKW], [JK] and the above remark, we have a unique harmonic map of annulus and disc type in $B(p, r) \subset N$ for a given boundary mapping in the class $H^{\frac{1}{2},2} \cap C^0$. Now we define

$$M^i \;\; := \;\; \{x^i \in \mathfrak{X}^i_{\text{mon}} \,|\, x^i \text{ preserves the orientation}\}.$$

Then M^i is complete, since the C^0-norm preserves monotonicity.

Moreover, let

$$\begin{aligned} \mathcal{S}(\Gamma_1, \Gamma_2) &= \{X \in H^{1,2} \cap C^0(\overline{A_\rho}, B(p, r)) |\, 0 < \rho < 1,\;\; X|_{C_i} \text{is weakly monotone}\}, \\ \mathcal{S}(\Gamma_i) &= \{X \in H^{1,2} \cap C^0(\overline{B}, B(p, r)) | X|_{\partial B} \text{ is weakly monotone}\}. \end{aligned}$$

II) We now investigate another significant condition for (N, h).

(C2) N is compact with nonpositive sectional curvature.

A compact Riemannian manifold is homogeneously regular and the condition of nonpositive sectional curvature implies $\pi_2(N) = 0$.

In order to define M^i, we first consider for $\rho \in (0, 1)$ the following

$$\tilde{G}_\rho := \{f \in H^{1,2} \cap C^0(\overline{A_\rho}, N) |\, f|_{C_i} \text{ is continuous, weakly monotone and onto } \Gamma_i\}.$$

We may take a continuous homotopy class, denoted by $\tilde{F}_\rho \subset \tilde{G}_\rho$, so that every two elements f, g in $\tilde{F}_\rho$ are continuously homotopic $f \sim g$ (not necessarily fixing the boundary parametrization). We further demand some relation $\tilde{F}_\rho \sim \tilde{F}_\sigma$ to hold for any $\rho, \sigma \in (0, 1)$. Precisely, for some $\tilde{f} \in \tilde{F}_\sigma$, $f \in \tilde{F}_\rho$ and some diffeomorphism $\tau^\rho_\sigma : [\sigma, 1] \to [\rho, 1]$, we require $\tilde{f}(r, \theta) = f(\tau^\rho_\sigma(r), \theta)$. Let $\tilde{F}_\rho$ be fixed. Then for any $\sigma \in (0, 1)$ we can find $\tilde{F}_\sigma$ with $\tilde{F}_\rho \sim \tilde{F}_\sigma$.

We now consider all possible $H^{1,2} \cap C^0$-extensions of disc type in N:

$$\mathcal{S}(\Gamma_i) := \{X \in H^{1,2} \cap C^0(\overline{B}, N) | X|_{\partial B} \text{ is weakly monotone onto } \Gamma_i\},$$

assuming that this set is not empty, for each $i = 1, 2$.

Lemma 2.1. *(i) For $X^1 \in \mathcal{S}(\Gamma_1)$ and $X^2 \in \mathcal{S}(\Gamma_2)$ there exists $f_\rho \in H^{1,2} \cap C^0(A_\rho, N)$ such that $f_\rho|_{C_1}(\cdot) = X^1|_{\partial B}(\cdot)$ and $f_\rho|_{C_\rho}(\cdot) = X^2|_{\partial B}(\frac{\cdot}{\rho})$, for $\rho \in (0,1)$.*

(ii) Moreover, there exists $\rho_0 \in (0,1)$ and a uniform positive constant C such that for some $f_\rho \in H^{1,2} \cap C^0(A_\rho, N)$, with $f_\rho|_{C_\rho}(\cdot) = X^2|_{\partial B}(\frac{\cdot}{\rho})$

$$E(f_\rho) \le C, \text{ for all } \rho \le \rho_0. \tag{2}$$

Proof. *(i)* For a given $\varepsilon > 0$, take $\sigma_i > 0$ with $\operatorname{osc}_{B_{\sigma_i}} X^i < \varepsilon$. Choose $\rho > 0$ with $\frac{\rho}{\sigma_2} < \sigma_1$, and let $\mathcal{H} : B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}} \to \mathbb{R}^k$ be harmonic with $X^1|_{\partial B_{\sigma_1}} - X^1(0)$ on ∂B_{σ_1} and $X^2|_{\partial B_{\sigma_2}} - X^2(0)$ on $\partial B_{\frac{\rho}{\sigma_2}}$. This implies $\|\mathcal{H}\|_{C^0} < \varepsilon$. Now let $g \in H^{1,2} \cap C^0(B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}}, N)$ with $X^1(0)$ on ∂B_{σ_1} and $X^2(0)$ on $\partial B_{\frac{\rho}{\sigma_2}}$.

Considering coordinate neighbourhoods for the submanifold $N \overset{\eta}{\hookrightarrow} \mathbb{R}^k$, we may take a finite covering of $f_\rho((\overline{A_\rho}))$, and by projection we obtain a smooth map $r : \mathcal{N}_\delta(f_\rho(\overline{A_\rho})) \to N$ with $r|_{\mathcal{N}_\delta(f_\rho(\overline{A_\rho})) \cap N} = Id$ for some $\delta > 0$, where $\mathcal{N}_\delta(\cdot)$ is δ-neighbourhood in $\mathbb{R}^k$. Setting $T(s,\theta) := (\frac{1}{s}\rho, \theta)$ in polar coordinates, we can define f_ρ with the desired properties:

$$f_\rho := \begin{cases} X^1|_{B \backslash B_{\sigma_1}} & , \text{ on } B \backslash B_{\sigma_1}, \\ r \circ (g + \mathcal{H}) & , \text{ on } B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}}, \\ X^2(T^{-1}(\cdot)) & , \text{ on } B_{\frac{\rho}{\sigma_2}} \backslash B_\rho. \end{cases} \tag{3}$$

(ii) The claim follows from the above construction, since $\frac{\rho}{\sigma_2} < \sigma_1$, $\rho \le \rho_0$ for some $\rho_0 > 0$. $\square$

Under the assumption that $\mathcal{S}(\Gamma_i) \neq \emptyset$, for given $\Gamma_i \in N$ we have an annulus-type-extension like that of (3), and we take homotopy classes which contain such an extension. From now on twiddles will be dropped.

Define

$$\mathcal{S}(\Gamma_1, \Gamma_2) \;\; := \;\; \{f \in F_\rho \,|\, 0 < \rho < 1\}, \tag{4}$$

as well as the two function spaces

$$\begin{aligned} M^1 &:= \{x^1(\cdot) = f|_{C_1}(\cdot),\ f \in \mathcal{S}(\Gamma_1,\Gamma_2) |\, x^1 \text{ is orientation preserving with degree 1}\}, \\ M^2 &:= \{x^2(\cdot) = f|_{C_\rho}(\cdot\rho),\ f \in \mathcal{S}(\Gamma_1,\Gamma_2) |\, x^2 \text{ is orientation preserving with degree 1}\}. \end{aligned}$$

For $x^i \in \mathcal{X}^i_{\text{mon}}$, $\mathcal{H}_\rho(x^1, x^2)$ denotes the unique $\mathbb{R}^k$-harmonic extension on A_ρ with $x^1(\cdot)$ on C_1 and $x^2(\frac{\cdot}{\rho})$ on C_ρ, while $\mathcal{H}(x)$ is the $\mathbb{R}^k$-harmonic extension of disc type with boundary $x \in \mathcal{X}^i_{\text{mon}}$.

Lemma 2.2. *(i) For each $x^i_0 \in M^i$, $i = 1,2$, there exists $\varepsilon(x^i_0) > 0$ such that*

$$\text{if } x^i \in \mathcal{X}^i_{mon} \text{ with } \|x^i - x^i_0\|_{\frac{1}{2},2;0} < \varepsilon, \text{ then } x^i \in M^i.$$

(ii) M^i *is complete with respect to* $\|\cdot\|_{\frac{1}{2},2;0}$.

Proof. *(i)* Let $f_\rho \in \tilde{F}_\rho$ with $f_\rho|_{C_1} = x_0^1$ and $f_\rho|_{C_\rho}(\cdot) = y^2(\frac{\cdot}{\rho})$ for some $y^2 \in M^2$.

We consider the smooth retraction $r : \mathcal{N}_\delta(f_\rho(\overline{A_\rho})) \to N$ as in the proof of Lemma 2.1. Let $\|x^i - x_0^i\|_{\frac{1}{2},2;0} < \varepsilon < \delta$. Then by Lemma 4.2 from [St3],

$$\begin{aligned}&\int_{A_\rho} |d(r(f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0)))|^2 d\omega \\ &\qquad \leq C(\|f_\rho\|_{C^0}, \varepsilon, N)\big(\int_{A_\rho} |df_\rho|^2 d\omega + \int_B |d\mathcal{H}(x^1 - x_0^1)|^2 d\omega\big) \leq C(\|f_\rho\|_{1,2;0}, \varepsilon, N).\end{aligned}$$

Now, let $H(t,\cdot) := (1-t)\mathcal{H}_\rho(x^1 - x_0^1, 0) : [0,1] \times A_\rho \to \mathbb{R}^k$ with $\|H\|_{C^0} < \varepsilon$ and $G : [0,1] \times A_\rho \to N$ with $G(t,\cdot) = f_\rho(\cdot)$ for all $t \in [0,1]$. Since $r(G+H) : [0,1] \times A_\rho \to N$ is a homotopy between f_ρ and $r(f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0))$, it follows $r(f_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0))(\sim f_\rho) \in \tilde{F}_\rho$, and $x^1 \in M^1$. Similarly, we can prove that $x^2 \in M^2$ if $\|x^2 - x_0^2\|_{\frac{1}{2},2;0} < \varepsilon'$ for some small $\varepsilon' > 0$.

(ii) A Cauchy sequence $\{x_n^i\} \subset M^i$ converges to $x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$, and for some n, $\|x_n^i - x^i\|_{C^0} < \varepsilon$. Considering $\mathcal{H}_\rho(x^1 - x_n^1, 0)$ and $g_\rho \in F_\rho$ with x_n^1 on C_1 and 0 on C_ρ, we can find a homotopy in N between g_ρ and $r(g_\rho + \mathcal{H}_\rho(x^1 - x_0^1, 0))$ as in *(i)*. We may also apply this argument to x^2. Note that x^i is weakly monotone, and hence $x^i \in M^i$, concluding the proof. $\square$

From the proof we easily conclude that the set of x^i's which possess annulus-type-extensions with uniform energy with respect to $\rho \leq \rho_0$ is an open and closed subset of $\mathcal{X}^i_{\text{mon}}$. Thus, it is a non-empty connected component of $\mathcal{X}^i_{\text{mon}}$ and must coincide with M^i, since M^i is a connected subset of $\mathcal{X}^i_{\text{mon}}$. Hence we obtain the following property.

Remark 2.2. *For each* $x^i \in M^i, i = 1, 2$, *there exist* $f_\rho \in \mathcal{S}(\Gamma_1, \Gamma_2)$ *and* $C > 0$ *with* $E(f_\rho) \leq C$ *for all* $\rho \leq \rho_0$ *and some* $\rho_0 \in (0,1)$. *Clearly, this result also holds for* $x^i \in M^i$ *if we assume (C1).*

For disc-type extensions of $x^i \in M^i$ the following lemmata will be useful.

Lemma 2.3. *Let* (N, h) *be a homogeneously regular manifold and* u *an absolutely continuous map of* $\partial B_r(x_0)$ *into* $N \ni x_0$ *with* $\int_0^{2\pi} |u'(\theta)|_h^2 d\theta \leq \frac{C'}{\pi}$. *Then there exists* $f \in H^{1,2}(B_r(x_0), N) \cap C^0(\overline{B_r(x_0)}, N)$ *with* $f|_{\partial B_r(x_0)} = u$ *and* $E_{B_r(x_0)}(f) \leq \frac{C''}{C'} \int_0^{2\pi} |u'(\theta)|_h^2 d\theta$, *where* C'', C' *are the constants defined by homogeneous regularity.*

Proof. See [Mo] Lemma 9.4.8 b). $\square$

Lemma 2.4. *Let* $f_\rho \in H^{1,2}(A_\rho, N)$, $0 < \rho < 1$. *For each* $\delta \in (\rho, 1)$ *there exists* $\tau \in (\delta, \sqrt{\delta})$ *with* $\int_0^{2\pi} \left|\frac{\partial f_\rho(\tau,\theta)}{\partial \theta}\right|_h^2 d\theta \leq \frac{4E(f_\rho)}{\ln \frac{1}{\delta}}$.

Proof. Similar to the proof of the Courant-Lebesgue Lemma. □

For $x^i \in M^i$, and given the choice of $\mathcal{S}(\Gamma_1, \Gamma_2)$, Remark 2.2 tells that we can find $f_\rho \in H^{1,2}(A_\rho, N)$ with boundary x^i such that $E(f_\rho) \le C$ for all $\rho \le \rho_0$. Then from Lemma 2.4 and Lemma 2.3, we have $g_\tau \in H^{1,2}(B_\tau, N)$ with boundary $f_\rho|_{\partial B_\tau}$ for some ρ. Together with g_τ and $f_\rho|_{B \setminus B_\tau}$, we obtain a map $X \in H^{1,2}(B, N)$ with boundary x^1. Similarly, we have $\tilde{X} \in H^{1,2}(B, N)$ with boundary x^2.

Moreover, the harmonic extension of disc type for each $x^i \in M^i$ in N is unique, independently of the choice of homotopy class $\mathcal{S}(\Gamma_1, \Gamma_2)$, because of the following well known fact.

Lemma 2.5. $\pi_2(N) = 0 \Leftrightarrow$ *Any* $h_0, h_1 \in C^0(B, N)$ *with* $h_0|_{\partial B} = h_1|_{\partial B}$ *are homotopic.*

On the other hand, using the construction (3) and the previous Lemma we can easily check that the traces of the elements in $\mathcal{S}(\Gamma_i)$ belong to M^i. From [ES], [Le], [Hm], we then have the following.

Remark 2.3. *(i) For $x^i \in M^i$, there exist a unique harmonic extension of disc type on B and of annulus type on A_ρ, $\rho \in (0,1)$.*

(ii) The elements of M^i are the traces of the elements of $\mathcal{S}(\Gamma_i)$.

III) Now let (N, h) and $\Gamma_i, i = 1, 2$ satisfy (C1) or (C2).

Observing $\partial B \cong \mathbb{R}/2\pi$, for a given oriented $y^i \in \mathcal{X}^i_{\text{mon}}$ there exists a weakly monotone map $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $w^i(\theta + 2\pi) = w^i(\theta) + 2\pi$ such that $y^i(\theta) = \gamma^i(\cos(w^i(\theta)), \sin(w^i(\theta))) =: \gamma^i \circ w^i(\theta)$. In addition $w^i = \tilde{w}^i + Id$ for some $\tilde{w}^i \in C^0(\partial B, \mathbb{R})$.

Denoting the Dirichlet integral by D and the $\mathbb{R}^k$-harmonic extension by $\mathcal{H}$, let

$$W^i_{\mathbb{R}^k} := \{w^i \in C^0(\mathbb{R}, \mathbb{R}) \,|\, w^i \text{ is weakly monotone}, w^i(\theta+2\pi) = w^i(\theta)+2\pi; D(\mathcal{H}(\gamma^i \circ w^i)) < \infty\}.$$

Clearly, $W^i_{\mathbb{R}^k}$ is convex (for further details, refer to [St1]).

Now take $x^i \in M^i$. Considering $w - w^i$ as a tangent vector along $\tilde{w}^i$, let

$$\mathcal{T}_{x^i} = \{d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w}^i) \,|\, w \in W^i_{\mathbb{R}^k} \text{ and } \gamma^i \circ w^i = x^i\}.$$

Note that $\mathcal{T}_{x^i}$ is convex in $T_{x^i} H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$, since $W^i_{\mathbb{R}^k}$ is convex. For $\xi = d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w}^i) \in \mathcal{T}_{x^i}$ we have that $\widetilde{\exp}_{x^i}\xi = \gamma^i(w)$, $\widetilde{\exp}$ denoting the exponential map with respect to the metric $\tilde{h}$.

If (C1) holds, then clearly $\widetilde{\exp}_{x^i}\xi \in M^i$ for $\xi \in \mathcal{T}_{x^i}$. For the case (C2), let us recall the proof of Lemma 2.2. Since N is compact, there exists $l_i > 0$, depending on γ^i, such that for any $x^i \in M^i$, $\widetilde{\exp}_{x^i}\xi \in M^i$, provided that $\|\xi\|_{\mathcal{T}_{x^i}} < l_i$.

The following set-up holds true in both cases (C1) and (C2).

Definition

(i) Let $\mathcal{M} := M^1 \times M^2 \times (0,1)$ with the product topology and $x := (x^1, x^2, \rho) \in \mathcal{M}$. Then the set $\mathcal{T}_x\mathcal{M} := \mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \times \mathbb{R}$ is convex.

Let $\mathcal{F}(x) = \mathcal{F}(x^1, x^2, \rho) = \mathcal{F}_\rho(x^1, x^2) : A_\rho \to N$ be the unique harmonic extension with x^1 on C^1 and $x^2(\frac{\cdot}{\rho})$ on C^2, and define

$$\begin{aligned} \mathcal{E} \ &: \ \mathcal{M} \longrightarrow \mathbb{R} \\ & x \longmapsto E(\mathcal{F}(x)) := \frac{1}{2}\int_{A_\rho} |d\mathcal{F}_\rho(x^1, x^2)|_h^2 d\omega. \end{aligned}$$

(ii) Define $\partial\mathcal{M} := M^1 \times M^2 \times \{0\}$, $\mathcal{T}_x\partial\mathcal{M} := \mathcal{T}_{x^1} \times \mathcal{T}_{x^2}$ and $\overline{\mathcal{M}} := \mathcal{M} \cup \partial\mathcal{M}$.

Let $\mathcal{F}^i(x^i) : A_\rho \to N$ be the unique harmonic extension with boundary x^i, for $x = (x^1, x^2, 0) \in \partial\mathcal{M}$, and define

$$\mathcal{E}(x) := E(\mathcal{F}^1(x^1)) + E(\mathcal{F}^2(x^2)).$$

2.3 Harmonic extension operators

Let $\Omega = A_\rho$ or $\Omega = B$. A weak Jacobi field $\mathbf{J}$ with boundary ξ along a harmonic function f is a weak solution of

$$\int_\Omega \langle \nabla \mathbf{J}, \nabla X\rangle + \langle tr\, R(\mathbf{J}, df)df, X\rangle d\omega = 0,$$

for all X $\in H^{1,2}(\Omega, f^*TN)$ with $X|_{\partial\Omega} = \xi$. Weak Jacobi fields are natural candidate derivatives of the harmonic operators $\mathcal{F}_\rho$ and $\mathcal{F}^i$.

We have the following property of weak Jacobi fields, from [Ho].

Lemma 2.6. *The weak Jacobi field* $\mathbf{J}$ *with boundary* $\eta \in T_{x^i}H^{\frac{1}{2},2} \cap C^0$ *along a harmonic* $\mathcal{F}$ *with boundary* x^i *is well defined in the class* $H^{1,2}$ *and continuous up to the boundary. It satisfies*

$$\|\mathbf{J}_\mathcal{F}\|_{C^0} \le \|\mathbf{J}_\mathcal{F}|_{\partial\Omega}\|_{C^0}, \ \ \|\mathbf{J}_\mathcal{F}\|_{1,2;0} \le C(N, \|f\|_{1,2:0})\|\mathbf{J}_\mathcal{F}|_{\partial\Omega}\|_{\frac{1}{2},2;0}.$$

Now we can discuss the differentiability of harmonic extension operators.

Lemma 2.7. *The operators* $\mathcal{F}_\rho, \mathcal{F}^i$ *are partially differentiable in* x^1 *(resp.* x^2*) for variations in* $T_{x^1}H^{\frac{1}{2},2} \cap C^0$ *(resp.* $T_{x^2}H^{\frac{1}{2},2} \cap C^0$*). Their derivatives are continuous Jacobi field operators with respect to* x^1, x^2*.*

Proof. The proof reproduces an argument we shall explain in full detail in Lemma 3.1, cases (B), (C), and as such will not be anticipated here. Alternatively, one can follow the aforementioned [Ho]. □

3 The variational problem

3.1 Differentiability of $\mathcal{E}$ on $\overline{\mathcal{M}}$

Lemma 3.1. *The following hold:*

(A) *$\mathcal{E}$ is continuously partially differentiable in x^1, x^2 with respect to variations in $\mathcal{T}_{x^1}$, $\mathcal{T}_{x^2}$ and the derivatives are continuous on $M^1 \times M^2$.*

(B) *$\mathcal{E}$ is continuous with respect to $\rho \in [0,1)$, even uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$ independent of $x_0^i \in M^i$, $i = 1, 2$.*

(C) *The partial derivatives in x^1, x^2 are continuous with respect to $\rho \in [0,1)$, uniformly continuous on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$ independent of $x_0^i \in M^i$, $i = 1, 2$.*

(D) *$\mathcal{E}$ is differentiable with respect to $\rho \in (0,1)$.*

Proof. From now on, continuity will be understood in the sense of convergence of subsequences.

(A) The Dirichlet integral functional is in C^∞, so Lemma 2.7 guarantees that $\mathcal{E}$ is continuously partially differentiable with continuous partial derivatives on $M^1 \times M^2$.

Computation of the derivatives:

Let $x = (x^1, x^2, \rho) \in \mathcal{M}$, $\xi^1 \in \mathcal{T}_{x^1}$. By Lemma 2.2 there is a small $t_0 > 0$ such that $\widetilde{\exp}_{x^1}(t\xi^1) \in M^1$, $0 \le t \le t_0$. Thus,

$$\begin{aligned}
\langle \delta_{x^1}\mathcal{E}, \xi^1 \rangle &:= \frac{d}{dt}\Big|_{t=0} \mathcal{E}(\widetilde{\exp}_{x^1}(t\xi^1), x^2, \rho) \\
&= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla D_{x^1}\mathcal{F}_\rho(x^1, x^2)(\xi^1)\rangle_h d\omega \\
&= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega \qquad \text{(by Lemma 2.7)}, \qquad (5)
\end{aligned}$$

since by computation we obtain, with $\mathcal{F}_\rho(t) := \mathcal{F}_\rho\left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right)$,

$$\nabla_{\frac{d}{dt}} \left(\mathcal{F}^\alpha_{\rho,i}(t) dx^i \otimes \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}_\rho(t) \right) = \nabla \frac{d}{dt} \mathcal{F}_\rho\left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right) \left(= \nabla\left(D_{x^1}\mathcal{F}_\rho(x^1, x^2)(\xi^1)\right), t = 0\right).$$

For $\xi^2 \in \mathcal{T}_{x^2}$ Lemma 2.7 yields $\langle \delta_{x^2}\mathcal{E}, \xi^2\rangle = \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(0, \xi^2(\frac{\cdot}{\rho}))\rangle_h d\omega$. Similarly, for $x = (x^1, x^2, 0) \in \partial\mathcal{M}$, $\langle \delta_{x^i}\mathcal{E}, \xi^i\rangle = \int_B \langle d\mathcal{F}^i(x^i), \nabla \mathbf{J}_{\mathcal{F}^i}(\xi^i)\rangle_h d\omega, i = 1, 2$.

For (B) we shall split the proof into three sub-steps *B*-I), *B*-II), *B*-III). Similarly for (C) we shall have *C*-I), *C*-II), *C*-III).

B-I) The set-up.

The claim is that $\mathcal{E}$ is continuous when $\rho \to \rho_0$. Fixing $\rho_0 = 0$ is no great restriction, since the proof for $\rho_0 \in (0,1)$ carries over in an analogous, even easier, fashion. Taking $\rho_0 = 0$ translates our claim into

$$\int_{A_\rho} \left|d\mathcal{F}_\rho(x^1,x^2)\right|_h^2 d\omega \longrightarrow \int_B \left|d\mathcal{F}^1(x^1)\right|_h^2 d\omega + \int_B \left|d\mathcal{F}^2(x^2)\right|_h^2 d\omega \tag{6}$$

uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$ independent of $x_0^i \in M^i$, whenever $\rho \to 0$.

Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1,x^2)$ and $\mathcal{F}^i := \mathcal{F}^i(x^i), i = 1,2$. By Lemma 2.4, for each δ with $0 < \rho < \delta < 1$ there exists $\nu \in (\delta, \sqrt{\delta})$ such that

$$\int_0^{2\pi} \left|\frac{\partial \mathcal{F}_\rho(\nu,\theta)}{\partial\theta}\right|_h d\theta \le \sqrt{2\pi}\left(\int_0^{2\pi}\left|\frac{\partial \mathcal{F}_\rho(\nu,\theta)}{\partial\theta}\right|_h^2 d\theta\right)^{\frac{1}{2}} \le \frac{C}{\sqrt{|\ln\delta|}}. \tag{7}$$

Due to Remark 2.2, C is independent of $\rho \le \rho_0$, for some $\rho_0 \in (0,1)$.

By means of $\mathcal{F}_\rho$ we now construct two maps by setting

$$\begin{aligned} f_\nu : A_\nu &\longrightarrow N \qquad \text{with} \qquad f_\nu(re^{i\theta}) := \mathcal{F}_\rho(re^{i\theta}),\ \ re^{i\theta} \in A_\nu, \\ g_{\nu'} : A_{\nu'} &\longrightarrow N \qquad \text{with} \qquad g_{\nu'}(re^{i\theta}) := \mathcal{F}_\rho(T(re^{i\theta})),\ \ re^{i\theta} \in A_{\nu'}. \end{aligned} \tag{8}$$

The constants $\nu' := \frac{\rho}{\nu}$, $\nu \in (\delta, \sqrt{\delta})$ and $\delta \in (\rho, 1)$ satisfy the property (7) in the limit $\nu', \nu \to 0$ for $\rho \to 0$. (One can take for instance $\delta = \sqrt{\rho}$). The map $T(re^{i\theta}) = \frac{\rho}{r}e^{i\theta}$ goes from $A_{\nu'}$ to $B_\nu \backslash B_\rho$ surjectively. Then, f_ν and $g_{\nu'}$ are harmonic maps into N with $f_\nu|_{\partial B} = x^1$, $g_{\nu'}|_{\partial B} = x^2$ and $\operatorname{osc}_{\partial B_\nu} f_\nu \to 0$, $\operatorname{osc}_{\partial B_{\nu'}} g_{\nu'} \to 0$ as $\rho \to 0$. Moreover, since T is conformal, $E(\mathcal{F}_\rho) = E(\mathcal{F}_\rho|_{A_\nu}) + E(\mathcal{F}_\rho|_{B_\nu\backslash B_\rho}) = E(f_\nu) + E(g_{\nu'})$ by conformal invariance of the Dirichlet integral.

B-II) The convergence of $\{f_\nu\}$, $\{g_{\nu'}\}$ to $\mathcal{F}^i$.

We first investigate the modulus of continuity of harmonic maps $\{h_\nu\} : A_\nu \to N$ which converge uniformly (C^0-norm) on ∂B with $E(h_\nu) \le L$ for some $L > 0$, independent of $\nu \le \nu_0$ for some $\nu_0 \in (0,1)$. We shall only deal with the assumption (C2), because the argument can clearly be applied to the case (C1) as well.

Let $G_R := \overline{B_R(z)} \subset A_\nu$ for $\nu \le \tilde{\nu_0}$. If $z \in \partial B$, consider $G_R := \overline{B_R(z)} \cap \overline{A_\nu}$. Given $\varepsilon > 0$, by the Courant-Lebesgue Lemma there exists $\delta > 0$, independent of $\nu \le \nu_0$, such that the length of $h_\nu|_{\partial G_\delta}$ does not exceed $\min\{\frac{\varepsilon}{4}, \frac{i(N)}{4}\}$, $i(N) > 0$. Then $h_\nu|_{\partial G_\delta} \subset B(q,s)$ for some $q \in N, s \le \min\{\frac{\varepsilon}{2}, \frac{i(N)}{2}\}$. Observe that h_ν is continuous on ∂G_δ, and there exists an $H^{1,2}$-extension X of disc type, whose image is in $B(q,s)$ with $X|_{\partial B_\delta} = h_\nu|_{\partial B_\delta}$, by the same argument of Remark 2.1. Thus there exists a harmonic extension h' with $h'(G_\delta) \subset B(q,s) \subset B(q,\frac{\varepsilon}{2})$, by [HKW]. From Lemma 2.5, h' is homotopic to h on G_δ, and from the energy minimizing property of harmonic maps, $h_\nu|_{G_\delta} = h'$. Hence, the functions h_ν with $\nu \le \nu_0$ have the same modulus of continuity. Furthermore, if these mappings have the same boundary image, they are C^0-uniformly bounded on each relatively compact domain.

Now apply the above result to $\{\mathcal{F}_\rho,\ \rho \leq \rho_0\}$ in $\mathbb{R}^k$. For some $\rho_0 \in (0,1)$ then, the functions f_ν resp. $g_{\nu'}$ have the same modulus of continuity for all $\rho \in (0,\rho_0)$, and some subsequences, denoted again by f_ν resp. $g_{\nu'}$ are locally uniformly convergent.

Recall that our maps are continuous, so by localizing in both domain and image, harmonic functions, seen as solutions of Dirichlet problems, may be also regarded as weak solutions f of the following elliptic systems in local coordinate charts of N:

$$d_i d_i f^\alpha = -\Gamma^\alpha_{\beta\gamma} d_i f^\beta d_i f^\gamma =: G^\alpha(\cdot, f(\cdot), df(\cdot)). \tag{9}$$

We can take the same coordinate charts for the image of $\{f_\nu\}_{\nu \leq \nu_0}$ and $\{g_{\nu'}\}_{\nu' \leq \nu'_0}$, where $\nu_0 := \nu(\rho_0)$, $\nu'_0 := \nu'(\rho_0)$, to the effect that we have the same weak solution system for (9). Moreover, since $h_{\alpha\beta}$ and $\Gamma^\alpha_{\beta\gamma}$ are smooth, the structure constants of the weak systems (see [Jo] section 8.5) are independent of $\rho \leq \rho_0$.

Now consider $K^\sigma_\sigma = \{\sigma \leq |z| \leq 1-\sigma\}$, $\sigma \in (0,1)$. From the regularity theory of [LU] and [Jo](section 8.5) and by the covering argument, there exists $C \in \mathbb{R}$ such that $\left\| f_\nu|_{K^\sigma_\sigma} \right\|_{H^{4,2}} \leq C$ for all $\nu \in (0,\nu_0)$. Hence the Sobolev's embedding theorem implies that for some sequence $\{\rho_i\} \subset (0,1)$, $\lim_{\rho_i \to 0} f_{\nu(\rho_i)}|_{K^\sigma_\sigma} = f'$ in $C^2(K^\sigma_\sigma, \mathbb{R}^n)$, with $\tau_h(f') = 0$ in K^σ_σ.

For $\sigma := \frac{1}{n}$, we choose a sequence $\{f_{\nu(\rho_{n,i})}\}$ as above such that $\{\rho_{n+1,i}\}$ is a subsequence of $\{\rho_{n,i}\}$. By diagonalizing we obtain a subsequence $\{f_{\nu(\rho_{n,n})}\}, n \geq n_0$ which converges locally to f' in the C^2-norm, so f' is harmonic on $B \backslash (\partial B \cup \{0\})$.

On the other hand $f_\nu|_{\partial B} = x^1$ for all ν, and the f_ν's converge uniformly to f' in a compact neighbourhood of ∂B. Thus, f' is continuous on $\overline{B} \backslash \{0\}$ with $f'|_{\partial B} = x^1$. Also observe that $\operatorname{osc}_{\partial B_r} f' \to 0$ as $r \to 0$, by construction.

For each compact $K \subset B \backslash \{0\}$, $\int_K |df'|^2 d\omega = \lim_{\rho_i \to 0} \int_K |df_{\nu(\rho_i)}|^2 \leq L$, with L independent of K. Thus, $f' \in H^{1,2}(B \backslash \{0\}, N)$, and f' can be extended to a weakly harmonic map on B ([Jo] Lemma 8.4.5, see also [SkU], [Grü]). Thus, f' can be considered weakly harmonic and $f' \in C^0(\overline{B}, N) \cap C^2(B, N)$ with $f'|_{\partial B} = x^1$, so uniqueness forces $f' = \mathcal{F}^1(x^1)$.

Similar results hold for $g_{\nu'}$.

B-III) The convergence of the energy.

We consider $\eta \circ f$, and denote it again by $f := (f^a)_{a=1,\cdots,k} \in H^{1,2}(\Omega, \mathbb{R}^k)$ for obvious reasons.

Since η is isometric, for $f := (f^\alpha)_{\alpha=1,\cdots,n} \in H^{1,2}(\Omega, N)$ we have $\int_\Omega |d(f^\alpha)|^2_h d\omega = \int_\Omega |d(f^a)|^2_{\mathbb{R}^k} d\omega$. A harmonic map $f \in H^{1,2}(\Omega, N)$ satisfies

$$\int_\Omega (\langle df, d\psi \rangle - \langle II \circ f(df, df), \psi \rangle) dw = 0 \tag{10}$$

for any $\psi \in H^{1,2}_0 \cap C^0(\Omega, \mathbb{R}^k)$, where II is the second fundamental form of η.

Set $K_\sigma = \{\sigma \leq |z| \leq 1\}, \sigma > 0$ and we consider $\mathbb{R}^k$-harmonic maps H_ν and $\widetilde{H}_\nu$ on K_σ with $H_\nu|_{\partial K_\sigma} = f_\nu|_{\partial K_\sigma}$ and $\widetilde{H}_\nu|_{\partial K_\sigma} = \mathcal{F}^1|_{\partial K_\sigma}$, where $\nu \in (0,\sigma)$. Let $H : B \to \mathbb{R}^k$ be harmonic with $H|_{\partial B} = H_\nu|_{\partial B} = \widetilde{H}_\nu|_{\partial B} = x^1$. Then $\{H_\nu\}, \{\widetilde{H}_\nu\}$ have the same modulus of continuity up to

∂B, and we have $\|H_\nu - H\|_{C^0;K_\sigma} \to 0, \quad \|\widetilde{H}_\nu - H\|_{C^0;K_\sigma} \to 0 \quad \text{as } \nu \to 0$. Furthermore, for $X_\nu := (f_\nu - \mathcal{F}^1) + (H_\nu - \widetilde{H}_\nu) \in H_0^{1,2} \cap C^0(K_\sigma, \mathbb{R}^k)$, we obtain

$$\|X_\nu\|_{(C^0;K_\sigma)} \leq \|f_\nu - \mathcal{F}^1\|_{C^0;K_\sigma} + \|H_\nu - H\|_{C^0;K_\sigma} + \|H - \widetilde{H}_\nu\|_{C^0;K_\sigma} \to 0 \quad \text{as } \nu \to 0.$$

Now consider

$$\begin{aligned}
&\int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), d(f_\nu - \mathcal{F}^1)\rangle d\omega \\
&\qquad = \underbrace{\int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), dX_\nu\rangle d\omega}_{:=I} - \underbrace{\int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), d(H_\nu - \widetilde{H}_\nu)\rangle d\omega}_{:=II}.
\end{aligned}$$

When $\nu \to 0$

$$\begin{aligned}
|I| &\leq \left|\int_{K_\sigma} \langle II \circ f_\nu(df_\nu, df_\nu), X_\nu\rangle d\omega\right| + \left|\int_{K_\sigma} \langle II \circ (d\mathcal{F}^1, d\mathcal{F}^1), X_\nu\rangle d\omega\right| \\
&= C(\|f_\nu\|_{1,2;0}), \|\mathcal{F}^1\|_{1,2;0})\|X_\nu\|_{(C^0;K_\sigma)} \to 0 \qquad (11)
\end{aligned}$$

from (10). Moreover, since $H_\nu - \widetilde{H}_\nu$ is harmonic in $\mathbb{R}^k$,

$$|II| \leq \int_{\partial B_\sigma} \left|\partial_r(H_\nu - \widetilde{H}_\nu)\right| d\omega \|f_\nu - \mathcal{F}^1\|_{C^0;K_\sigma} \to 0 \quad \text{as } \nu \to 0. \qquad (12)$$

Thus $\int_{K_\sigma} |d(f_\nu - \mathcal{F}^1)|^2 d\omega \to 0$, and $\int_{K_\sigma} |df_\nu|^2 d\omega \to \int_{K_\sigma} \left|d\mathcal{F}^1\right|^2 d\omega$, for any K_σ. Since $\int_{B_\sigma} |d\mathcal{F}^1|^2 d\omega \to 0$ as $\sigma \to 0$, we obtain $\int_{A_\nu} |df_\nu|^2 d\omega \to \int_B \left|d\mathcal{F}^1\right|^2 d\omega$ as $\nu \to 0$. Similarly $\int_{A_{\nu'}} |dg'_\nu|^2 d\omega \to \int_B \left|d\mathcal{F}^2\right|^2 d\omega \quad \text{as } \nu' \to 0$.

Now to the uniform convergence on $\mathcal{N}_\varepsilon(x_0^i)$. Replace $f(\overline{A_\rho})$ by $\overline{B(p,r)}$(for (C1)) or N (for (C2)) in the proof of Lemma 2.2. Then, $\|\mathcal{F}_\rho(x^1, x^2)\|_{H^{1,2}} \leq C$ uniformly on $\mathcal{N}_\varepsilon(x_0^i)$, where the constant C depends on x_0^i, while ε does not. The convergence in (11), (12) is uniform on $\mathcal{N}_\varepsilon(x_0^i)$. The proof of *(B)* is eventually completed.

C-I)The set-up.

We must show that for $x^i \in M^i$ and $\xi^i \in \mathcal{T}_{x^i}$,

$$\langle \delta_{x^i}\mathcal{E}_\rho, \xi^i\rangle \quad \longrightarrow \quad \langle \delta_{x^i}\mathcal{E}, \xi^i\rangle \quad \text{uniformly on } \mathcal{N}_\varepsilon(x_0^i) \subset M^i, i = 1, 2 \text{ as } \rho \to 0.$$

It suffices to show the assertion for $i = 1$. We know that

$$\begin{aligned}
\langle \delta_{x^1}\mathcal{E}_\rho, \xi^1\rangle &= \int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla\mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega + \int_{B_{\nu(\rho)}\setminus B_\rho} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla\mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega \\
&= \int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1, x^2), \nabla\mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega + \int_{A_{\nu'}} \langle dg_{\nu'}, \nabla\mathbf{J}_{g_{\nu'}}(0, \zeta_{\nu'})\rangle d\omega,
\end{aligned}$$

where $g_{\nu'}(\cdot) = \mathcal{F}_\rho \circ T(\cdot)$ and $\zeta_{\nu'}(\nu' e^{i\theta}) = \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)(\nu e^{i\theta})$ with $\nu' := \frac{\rho}{\nu(\rho)}$. Observe $\mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \circ T$ is a Jacobi field along $g_{\nu'}$, by the conformal property of T.

C-II) The convergence of Jacobi fields.

First, let $V_\nu := \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)|_{A_\nu} = v_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ f_\nu$, for which we will show the existence of a $\nu_0 \in (0,1)$ giving

$$\|DV_\nu\|_2^2 := \int_{A_\nu} h_{\alpha\beta} \circ f_\nu v^\alpha_{\nu,i} v^\beta_{\nu,i} d\omega \le C \text{ for all } \nu \in (0, \nu_0). \tag{13}$$

By direct computation $\|DV_\nu\|_2^2 \le CE(V_\nu) + C(N, \|V_\nu\|_{C^0}, \|f_\nu\|_{C^0}, E(f_\nu))$. Since Lemma 2.6 yields $\|V_\nu\|_{C^0} \le \|\xi^1_\nu\|_{C^0}$, we only need to show that

$$E(V_\nu) := \int_{A_\nu} |\nabla^{f_\nu} V_\nu|^2 d\omega \le C, \quad \nu \in (0, \nu_0). \tag{14}$$

Let $X_\nu := x^\alpha_\nu \frac{\partial}{\partial y^\alpha} \circ f_\nu \in H^{1,2}(A_\nu, f_\nu^* TN)$, where $x^\alpha_\nu(z) := v^\alpha_{2\nu_0}(\tau^{2\nu_0}_{\nu_0}(z))$, $\nu_0 \le |z| \le 1$ (see section 2.2 for the definition of $\tau^{2\nu_0}_{\nu_0}$) and $x^\alpha_\nu(z) := 0$, $\nu \le |z| \le \nu_0$. Clearly, $\|DX_\nu\|_2^2 \le C(\nu_0, N)\|DV_{2\nu_0}\|_2^2$ for all $\nu \le \nu_0$.

By the minimality property of Jacobi fields and Young's inequality,

$$\begin{aligned}
&\int_{A_\nu} (|\nabla^{f_\nu}(V_\nu)|^2 - \langle trR(df_\nu, V_\nu) df_\nu, V_\nu\rangle) d\omega \le \int_{A_\nu} (|\nabla^{f_\nu}(X_\nu)|^2 - \langle trR(df_\nu, X_\nu) df_\nu, X_\nu\rangle) d\omega \\
&\le \int_{A_\nu} h_{\alpha\beta} \circ f_\nu x^\alpha_{\nu,i} x^\beta_{\nu,i} d\omega + \varepsilon \int_{A_\nu} |x^\alpha_{\nu,i} \frac{\partial}{\partial y^\alpha} \circ f_\nu|_h^2 d\omega + \varepsilon^{-1} \int_{A_\nu} |x^\gamma_\nu f^\delta_{,i} \Gamma^\beta_{\gamma\delta} \circ f_\nu \frac{\partial}{\partial y^\beta} \circ f_\nu|_h^2 d\omega \\
&\quad + \int_{A_\nu} h_{\alpha\beta} \circ f_\nu x^\gamma_\nu x^\lambda_\nu f^\delta_{,i} f^\mu_{,i} \Gamma^\alpha_{\gamma\delta} \circ f_\nu \Gamma^\beta_{\lambda\mu} \circ f_\nu d\omega - \int_{A_\nu} \langle trR(df_\nu, X_\nu) df_\nu, X_\nu\rangle d\omega \\
&\le C(N, \varepsilon, \|f_\nu\|_{C^0}, E(f_\nu), \|V_{2\nu_0}\|_{C^0}, \|DV_{2\nu_0}\|_2^2).
\end{aligned}$$

But $E(V_\nu) \le C$, $\nu \in (0, \nu_0)$, since

$$\int_{A_\nu} \langle trR(df_\nu, V_\nu) df_\nu, V_\nu\rangle) d\omega \le C(N, \|f_\nu\|_{C^0}, E(f_\nu), \|\xi^1\|_{C^0}).$$

Therefore we have (13), and this means that $\{(v^\alpha_\nu) | \nu \le \nu_0\}_{\alpha=1,\cdots,n}$ has the same modulus of continuity, see the argument in *B*-III) and Lemma 2.6.

With the same charts as in *(B)*, $(v^\alpha_{\nu(\rho)}) \in \mathbb{R}^n$, $\nu \le \nu_0$ are weak solutions of the Jacobi fields system with uniformly bounded energy and same modulus of continuity on $K_\sigma = \{\sigma \le |z| \le 1\}$, with $\sigma > 0$ for small ρ, again by Lemma 2.6. Just as in *(B)*, $\{V_\nu\}$ converges to the Jacobi field along $\mathcal{F}^1|_{B\setminus\{0\}}$ with boundary ξ^1, and for $\mathbf{J}_{\mathcal{F}^1}(\xi^1) =: w^\alpha \frac{\partial}{\partial y^\beta} \circ \mathcal{F}^1$, we have

$$\|(v^\alpha_\nu(z)) - (w^\alpha(z))\|_{C^0; K_\sigma} \to 0, \;\; \|(v^\alpha_\nu(z)) - (w^\alpha(z))\|_{C^2; K} \to 0, \quad \text{as } \nu (\text{or } \rho) \to 0,$$

on any compact $K \subset B\setminus\{0\}$.

C-III) The convergence of derivatives.

Taking K_σ as above, we denote $f_\nu|_{K_\sigma}$ and $\mathcal{F}^1|_{K_\sigma}$ by f_ν and $\mathcal{F}^1$, respectively.

Note that $\exp_{\mathcal{F}^1} : \mathcal{U}(0) \to H^{1,2} \cap C^0(K_\sigma, N)$ is a diffeomorphism on some neighbourhood $\mathcal{U}(0) \in H^{1,2} \cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$, because $d(\exp_{\mathcal{F}^1})_0 = Id$. Moreover, $\|f_\nu - \mathcal{F}^1|_{K_\sigma}\|_{H^{1,2}\cap C^0} \to 0$ as $\nu \to 0$, so there exists $\xi_\nu \in H^{1,2} \cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$ for small $\nu > 0$ with $\exp_{\mathcal{F}^1} \xi_\nu = f_\nu$.

The mapping $\xi \mapsto d\exp_{\mathcal{F}^1,\xi}$ depends smoothly on $\xi_\nu \in T_{\mathcal{F}^1} H^{1,2} \cap C^0(K_\sigma, N)$, so $d\exp_{\mathcal{F}^1,\xi_\nu} \to Id$ in $H^{1,2} \cap C^0(K_\sigma)$, since $\xi_\nu \to 0$ in $H^{1,2} \cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$ as $\nu \to 0$. For $W_\nu := w_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1 := d\exp^{-1}_{\mathcal{F}^1,\xi_\nu}(V_\nu)$ we have $\|w_\nu^\alpha(z) - w^\alpha(z)\|_{C^0;K_\sigma} \to 0$ by C-II). Moreover, $d\mathcal{F}^1 \to df_\nu$ in L^2, thus $\int_{K_\sigma} |d\exp_{\mathcal{F}^1,\xi_\nu}(d\mathcal{F}^1) - df_\nu|^2 d\omega \to 0$.

We next observe, for $\nabla^{\mathcal{F}^1} W_\nu = (w^\alpha_{\nu,i} + w^\gamma_\nu (\mathcal{F}^1)^\beta_{,i} \Gamma^\alpha_{\beta\gamma}(\mathcal{F}^1)) dz^i \otimes \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1$, that

$$\int_{K_\sigma} |d\exp_{\mathcal{F}^1,\xi_\nu}(\nabla^{\mathcal{F}^1} W_\nu) - \nabla^{f_\nu} V_\nu|^2 d\omega \to 0 \ \text{ as } \ \nu \to 0, \tag{15}$$

since $\|\mathcal{F}^1 - f_\nu\|_{1,2;0} \to 0$, $d\exp_{\mathcal{F}^1,\xi_\nu} \to Id$ in C^0, $\partial_i(d\exp_{\mathcal{F}^1,\xi_\nu}) \to \partial_i(Id) = 0$ in L^2.

Thus, for $X_\nu, Y_\nu \in H^{1,2} \cap C^0(K_\sigma, T^*M \otimes f_\nu^* TN)$ with $\int_{K_\sigma} |X_\nu|^2 d\omega \to 0$, $\int_{K_\sigma} |Y_\nu|^2 d\omega \to 0$,

$$d\exp_{\mathcal{F}^1,\xi_\nu}(d\mathcal{F}^1) = df_\nu + X_\nu, \ \ d\exp_{\mathcal{F}^1,\xi_\nu}(\nabla^{\mathcal{F}^1} W_\nu) = \nabla^{f_\nu} V_\nu + Y_\nu.$$

Gauß lemma prescribes that $\langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} W_\nu \rangle_h = \langle df_\nu + X_\nu, \nabla^{f_\nu} V_\nu + Y_\nu \rangle_h$. Thus the Hölder inequality and (14) give

$$\begin{aligned}
&\int_{K_\sigma} \Big(\langle df_\nu, \nabla^{df_\nu} V_\nu \rangle_h - \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1) \rangle_h \Big) d\omega \\
&\qquad = \int_{K_\sigma} \Big(\langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} W_\nu \rangle_h - \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1) \rangle_h \Big) d\omega + o(1) \\
&\qquad \leq E(d\mathcal{F}^1) \|\nabla^{\mathcal{F}^1} W_\nu - \nabla^{\mathcal{F}^1} \mathbf{J}_{\mathcal{F}^1}(\xi^1)\|_{L^2;K_\sigma} + o(1).
\end{aligned} \tag{16}$$

In order to estimate the last term, consider $A_\nu := a_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1$ and $A := a^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1$ such that $d\eta\big(a_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1\big)$ and $d\eta\big(a^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}^1\big)$ are harmonic in $\mathbb{R}^k$ with $A_\nu|_{\partial K_\sigma} = W_\nu|_{\partial K_\sigma}$, $A|_{\partial K_\sigma} = W|_{\partial K_\sigma}$, for $W := \mathbf{J}_{\mathcal{F}^1}(\xi^1)$. Clearly, $\|d\eta(A_\nu - A)\|_{1,2;0} \to 0$.

Now, consider a test vector field $Z_\nu := W_\nu - W - A_\nu + A \in H^{1,2}_0 \cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$. Observing that W and V_ν are Jacobi fields along $\mathcal{F}^1|_{K_\sigma}$ and $f_\nu|_{K_\sigma}$ respectively, we have

$$\begin{aligned}
&\int_{K_\sigma} \langle \nabla^{\mathcal{F}^1}(W_\nu - W), \nabla^{\mathcal{F}^1} Z_\nu \rangle_h d\omega \\
&= \int_{K_\sigma} \{ \langle \nabla^{\mathcal{F}^1} W_\nu, \nabla^{\mathcal{F}^1} Z_\nu \rangle_h - \langle trR \circ \mathcal{F}^1(W, d\mathcal{F}^1) d\mathcal{F}^1, Z_\nu \rangle_h \\
&\qquad - \langle \nabla^{f_\nu} V_\nu, \nabla^{f_\nu}(L_\nu(Z_\nu)) \rangle_h + \langle trR \circ f_\nu(V_\nu, df_\nu) df_\nu, (L_\nu(Z_\nu)) \rangle_h \} d\omega \\
&= \int_{K_\sigma} \{ \langle \nabla^{\mathcal{F}^1} W_\nu, \nabla^{\mathcal{F}^1} Z_\nu \rangle_h - \langle trR \circ \mathcal{F}^1(W, d\mathcal{F}^1) d\mathcal{F}^1, Z_\nu \rangle_h \\
&\qquad - \langle \nabla^{\mathcal{F}^1} L_\nu^{-1}(V_\nu), \nabla^{\mathcal{F}^1} Z_\nu \rangle_h + \langle trR \circ f_\nu(V_\nu, df_\nu) df_\nu, (L_\nu(Z_\nu)) \rangle_h \} d\omega + o(1)
\end{aligned}$$

with $L_\nu := d\exp_{\mathcal{F}^1,\xi_\nu}$. This expression converges to 0 as $\nu \to 0$, since $L_\nu^{-1}(V_\nu) = W_\nu$, $\|Z_\nu\|_{C^0;K_\sigma} \to 0$ and $\|\mathcal{F}^1\|_{1,2;0}, \|W\|_{C^0}, \|f_\nu\|_{1,2;0}, \|V_\nu\|_{C^0} < C$ for all $\nu \in (0,\nu_0)$.

Moreover, $\int_{K_\sigma} |\nabla^{\mathcal{F}^1}(A_\nu - A)|_h^2 d\omega \to 0$, since $\|d\eta(A_\nu - A)\|_{C^0} \to 0$ and because of (1). Thus, (16) converges to 0 for each $\sigma \in (0,1)$. Now let $\sigma \to 0$. Then

$$\int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1,0)\rangle_h d\omega \to \int_B \langle d\mathcal{F}^1(x^1), \nabla \mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h d\omega, \quad \rho \to 0,$$

since $\int_{B_\sigma} \langle d\mathcal{F}^1(x^1), \nabla \mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h d\omega \to 0$ as $\sigma \to 0$.

In a similar way, $\int_{A_{\nu'(\rho)}} \langle dg_{\nu'}, \nabla \mathbf{J}_{g_{\nu'}}(0,\zeta_{\nu'})\rangle d\omega \to \int_B \langle d\mathcal{F}^2(x^2), \nabla \mathbf{J}_{\mathcal{F}^2}(0)\rangle_h d\omega = 0$.

The uniform convergence on $\mathcal{N}_\varepsilon(x_0^i)$ is clear.

In this manner we could also show that $\delta_{x^1}\mathcal{E}_\rho, \delta_{x^2}\mathcal{E}_\rho$ are continuous with respect to $\rho \in (0,1)$, and uniformly continuous on $\mathcal{N}_\varepsilon(x_0^i)$. This concludes part *(C)*.

(D) Along the lines of [St3], the differential form

$$\frac{\partial}{\partial t}|_{t=\rho}\mathcal{E}(x^1,x^2,t) = \int_0^{2\pi}\int_\rho^1 \left[|\partial_r \mathcal{F}_\rho|^2 - \frac{1}{r^2}|\partial_\theta \mathcal{F}_\rho|^2\right]\frac{1}{1-\rho}\,dr\,d\theta \tag{17}$$

proves *(D)*, bringing to an end the proof of Lemma 3.1. □

3.2 Critical points of $\mathcal{E}$

For given Jordan curves $\Gamma_1, \Gamma_2, \Gamma$ in (N,h) with $\mathrm{dist}(\Gamma_1,\Gamma_2) > 0$, we consider the Plateau problems $\mathcal{P}(\Gamma_1,\Gamma_2)$ and $\mathcal{P}(\Gamma)$.

We define for $x = (x^1,x^2,\rho) \in \overline{\mathcal{M}}$,

$$\begin{aligned} g_i(x) &:= \sup_{\substack{\xi^i \in \mathcal{T}_{x^i} \\ \|\xi^i\| < l_i}} (-\langle \delta_{x^i}\mathcal{E}, \xi^i\rangle), \qquad i = 1,2, \\ g_3(x) &:= \begin{cases} |\rho\cdot\partial_\rho\mathcal{E}| & , \quad \rho > 0 \\ 0 & , \quad \rho = 0, \end{cases} \\ g(x) &:= \Sigma_{j=1}^3 g_j(x). \end{aligned} \tag{18}$$

In the definition of l_i of section 2.2, we can clearly require that $l_i \leq \{1, i_{\tilde{h}}(\Gamma_i)\}$. Note that $g_j \geq 0, j = 1,2,3$, because $g_i(x) < 0,\ i = 1,2$ would imply $\langle \delta_{x^i}\mathcal{E}, \xi^i\rangle \geq \sigma > 0$ for all $\xi^i \in \mathcal{T}_{x^i}$ with $\|\xi^i\| < l_i$. Since $\mathcal{T}_{x^i}$ is convex, $\langle \delta_{x^i}\mathcal{E}, t\xi^i\rangle = t\sigma \geq \sigma$, $t \in [0,1]$, a contradiction. Clearly, $g_3(x) \geq 0$. Now we are ready to define the critical points of $\mathcal{E}$.

Definition $x \in \overline{\mathcal{M}}$ is a critical point of $\mathcal{E}$ if $g(x) = 0$, i.e. $g_j = 0, j = 1,2,3$.

Lemma 3.2. *The functions g_j are continuous, $j = 1,2,3$. In particular, as $\rho \to \rho_0 \in [0,1)$, $g_j(x^1,x^2,\rho)$ converges uniformly to $g_j(x^1,x^2,\rho_0)$ on $\mathcal{N}_\varepsilon(x^i)$, $i = 1,2$, for some small $\varepsilon > 0$.*

Proof. The uniform convergence of g_i follows immediately from the uniform convergence of $\delta_{x^i}\mathcal{E}$, see Lemma 3.1 *(C)*.

Let $\{x_n\} = \{(x_n^1, x_n^2, \rho_n)\} \subset \overline{\mathcal{M}}$ strongly converge to $x = (x^1, x^2, \rho)$. From the above, $g_i(x_n^1, x_n^2, \rho_n) \to g_i(x_n^1, x_n^2, \rho)$ uniformly on $\{n \geq n_0\}$.

Let $\tilde{x}_n := (x_n^1, x_n^2, \rho)$ and $\widetilde{\exp}_{x_n^i}\xi_n^i = x^i$. Observe that $d\widetilde{\exp}_{x_n^i,\xi_n^i} \to Id$ in $H^{\frac{1}{2},2} \cap C^0$, hence for some t_0 independent of $n \geq n_0$, $\|t_0 d\widetilde{\exp}_{x_n^i,\xi_n^i}(\eta_n^i)\|_{\mathcal{T}_{x^i}} < l_i$ if $\|\eta_n^i\|_{\mathcal{T}_{x_n^i}} < l_i$. Note that $\mathcal{T}_{x^i}$ is convex and contains zero.

Then by Lemma 3.1 *(A)*, for given $\delta > 0$ there exist $t_0(\delta)$ and $n_0(\delta)$ as above such that for each $\|\eta_n^i\|_{\mathcal{T}_{x_n^i}} < l_i$ with $n \geq n_0(\delta)$,

$$\begin{aligned}
-\langle \delta_{x^i}\mathcal{E}(\tilde{x}_n), \eta_n^i\rangle &\leq -\langle \delta_{x^i}\mathcal{E}(x), d\widetilde{\exp}_{x_n^i,\xi_n^i}(\eta_n^i)\rangle + \delta \\
&\leq -\langle \delta_{x^i}\mathcal{E}(x), t_0 d\widetilde{\exp}_{x_n^i,\xi_n^i}(\eta_n^i)\rangle + 2\delta \leq g_i(x) + 2\delta.
\end{aligned}$$

This implies $g_i(\tilde{x}_n) \leq g_i(x) + 2\delta$. On the other hand $g_i(x) \leq g_i(\tilde{x}_n) + 2\delta$, so $g_i(x_n^1, x_n^2, \rho) \to g_i(x^1, x^2, \rho)$ as $n \to \infty$.

Together with the above uniform convergence on $\mathcal{N}_\varepsilon(x^i)$ for $\rho_n \to \rho$, we infer the continuity of g_i, $i = 1, 2$. The continuity and uniform continuity of g_3 are easy consequences of the expression of $\frac{\partial}{\partial\rho}\mathcal{E}$. $\square$

Proposition 3.1. $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times [0, 1)$ *is a critical point of* $\mathcal{E}$ *if and only if* $\mathcal{F}_\rho(x^1, x^2)$ *(for* $\rho \in (0, 1)$*), resp.* $\mathcal{F}^i(x^i)$ *is a solution of* $\mathcal{P}(\Gamma_1, \Gamma_2)$*, resp.* $\mathcal{P}(\Gamma_i), i = 1, 2$.

Proof. (I) Let $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times [0, 1)$ be a critical point of $\mathcal{E}$. From [HKW] $\mathcal{F}$ is continuous up to the boundary. We must show that $\mathcal{F}_\rho(x^1, x^2)$(for $\rho > 0$) and $\mathcal{F}^i(x^i)$ are conformal. We will show this only for $\mathcal{F}_\rho(x^1, x^2)$, the other case being analogous.

For $x \in \mathcal{M}$ critical point of $\mathcal{E}$, we have $\mathcal{F}_\rho(x^1, x^2) \in H^{2,2}(A_\rho, \mathbb{R}^k)$ from Theorem A.1. The condition $\gamma^i \in C^3, i = 1, 2$ will be essential. Taking $\xi^1 \in \mathcal{T}_{x^1}$, and denoting $\mathcal{F}_\rho(x^1, x^2)$ and $\mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)$ by $\mathcal{F}_\rho$ and $\mathbf{J}_\rho$, we compute $\xi^1 \in \mathcal{T}_{x^1}$,

$$\begin{aligned}
\langle \delta_{x^1}\mathcal{E}, \xi^1\rangle &= \int_{A_\rho} \langle d\mathcal{F}_\rho, \nabla_{\frac{d}{dt}} d\mathcal{F}_\rho(\widetilde{\exp}_{x^1} t\xi^1, x^2)\big|_{t=0}\rangle_h d\omega = \int_{A_\rho} \langle \frac{\partial}{\partial z^i}\mathcal{F}_\rho, \nabla_{\frac{\partial}{\partial z^i}} \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega \\
&= \int_{A_\rho} \operatorname{div}(\langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho, \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h, \langle \frac{\partial}{\partial z^2}\mathcal{F}_\rho, \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h) d\omega \ \ (\text{since } \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^i}\mathcal{F}_\rho = 0) \\
&= \int_{\partial B} \langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho \vec{n}, \xi^1\rangle_h d\omega. \qquad (19)
\end{aligned}$$

The work in [St1] leads to the conformal property of $\mathcal{F}_\rho$.

(II) Let $\mathcal{F} := \mathcal{F}_\rho(x)$(resp. $\mathcal{F}^i(x^i)$) be a minimal surface of annulus (resp. disc) type. By [HH], $\mathcal{F} \in C^1(\overline{A_\rho}, N)$ (resp. $C^1(\overline{B}, N)$). Conformality implies $\frac{d\mathcal{F}}{d\vec{n}} \cdot \frac{d}{d\theta}x^i \equiv 0$, and (19) says that $g_1(x) = 0, g_2(x) = 0$. That $g_3(x) = 0$ follows from using (17) as well. $\square$

4 Unstable minimal surfaces

4.1 The Palais-Smale condition

By the conformal invariance of the energy function E, the Palais-Smale (PS) condition cannot be satisfied in the former setting for $\mathcal{E}$ (cf. [St1] Lemma I.4.1). Hence we need the normalization used in [St3]: With $P_k^i \in \Gamma_i$ fixed, $k = 1, 2, 3,\ i = 1, 2$, let

$$M^{i*} = \{x^i \in M^i : x^i(\cos\frac{2\pi(k-1)}{3}, \sin\frac{2\pi(k-1)}{3}) = P_k^i \in \Gamma_i,\ \ k = 1,\ 2,\ 3\}.$$

Now define

$$\begin{aligned} \mathcal{M}^* &= \{x = (x^1, x^2, \rho) \in \mathcal{M} : x^1(1,0) = P_1^1 \in \Gamma_1\}, \\ \partial\mathcal{M}^* &= \{x = (x^1, x^2, 0) \in \partial\mathcal{M} : x^i \in M^{i*}\}. \end{aligned}$$

Given $x \in \mathcal{M}^*$ and $x \in \partial\mathcal{M}^*$ we take the variations from $\mathcal{T}_x\mathcal{M} = \mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \times \mathbb{R}$ and $\mathcal{T}_x\partial\mathcal{M} = \mathcal{T}_{x^1} \times \mathcal{T}_{x^2}$ respectively, namely we use the original tangent spaces.

We consider the following topology:

- A neighbourhood $\mathcal{U}_\varepsilon(x_0)$ of $x_0 = (x_0^1, x_0^2, 0) \in \partial\mathcal{M}^*$ consists of all $x = (x^1, x^2, \rho) \in \overline{\mathcal{M}^*}$ such that $\rho < \varepsilon$ and for each $i = 1, 2$, $\inf_{\{\text{all}\ \ \sigma\}} \|\mathcal{F}^i(x^i) \circ \sigma - \mathcal{F}^i(x^i)\|_{1,2} < \varepsilon$, where σ is a conformal diffeomorphism of B.
- A sequence $\{x_n = (x_n^1, x_n^2, \rho_n)\} \subset \overline{\mathcal{M}}^*$ converges strongly to $x = (x^1, x^2, 0) \in \partial\mathcal{M}^*$, if all but finitely many x_n lie in $\mathcal{U}_\varepsilon(x)$, for any $\varepsilon > 0$.

With respect to this topology $g_j, j = 1, 2, 3$, are continuous and uniformly continuous as $\rho \to \rho_0 \in [0, 1)$ on some ε-neighborhood of (x^1, x^2), because of Lemma 3.2 and the invariance of the Dirichlet integral under conformal changes.

Proposition 4.1 (Palais-Smale condition). *Suppose $\{x_n\}$ is a sequence in $\overline{\mathcal{M}^*}$ such that $\mathcal{E}(x_n) \to \beta,\ \ g(x_n) \to 0$, as $n \to \infty$. Then there exists a subsequence of $\{x_n\}$ which converges strongly to a critical point of $\mathcal{E}$ in $\overline{\mathcal{M}^*}$.*

Proof. We prove this for the case $\{x_n\} \subset \mathcal{M}^*$ with $0 < \rho_n < 1$, $\mathcal{E}(x_n) \to \beta$, $g_j(x_n) \to 0$. If $\{x_n\} \subset \partial\mathcal{M}^*$, the proof is similar. We may additionally suppose that $\rho_n \to \rho$.

Note that ρ cannot be 1, i.e. $0 \le \rho < 1$, because for any $x = (x^1, x^2, \rho) \in \mathcal{M}$, $\frac{\rho}{1-\rho} \le c\mathcal{E}(x)$, since $0 < \mathrm{dist}(\Gamma_1, \Gamma_2)$. More on this can be found in [St3], Lemma 4.10.

Clearly $\int_{A_\rho} |d\eta \circ \mathcal{F}_\rho(x^1, x^2)|^2 d\omega \ \ge\ \int_{A_\rho} |d\mathcal{H}_\rho(x^1, x^2)|^2 d\omega \ \ge\ C(\rho)\Sigma_i \int_B |d\mathcal{H}(x^i)|^2 d\omega$. Thus Proposition II.2.2 of [St1] guarantees that for some subsequence $\{w_n^i\}$ with $\gamma^i(w_n^i) = x_n^i$, we either have $\|w_n^i - w^i\|_{C^0} \to 0$ with $\gamma^i \circ w^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, \Gamma_i)$, or $x_n^i = \gamma^i \circ w_n^i \to \text{const.} = a_i \in \Gamma_i$ in $L^1(\partial B)$. Therefore we have to distinguish four main cases, each divided in sub-steps.

(case 1) Let $\rho \in (0, 1)$ and $\|w_n^i - w^i\|_{C^0} \to 0$, i.e. $\|x_n^i - x^i\|_{C^0} \to 0, x^i \in H^{\frac{1}{2},2} \cap C^0,\ \ i = 1, 2$.

First, $\gamma^i(w_n^i(\theta)) - \gamma^i(w^i(\theta)) = \underbrace{d\gamma^i(w_n^i(\theta))(w_n^i(\theta) - w^i(\theta))}_{=:I_n^i} - \underbrace{\int_{w^i(\theta)}^{w_n^i(\theta)} \int_{w'}^{w^i(\theta)} d^2\gamma^i(w'')dw''dw'}_{=:II_n^i}.$

In addition, $\int_{A_\rho} |d\mathcal{H}_\rho(II_n^1, II_n^2)|^2 d\omega \leq C(\rho)(\|\mathcal{H}(II_n^1)\|_{1,2;0}^2 + \|\mathcal{H}(II_n^2)\|_{1,2;0}^2) \to 0$, as $n \to \infty$, since $\|II_n^i\|_{\frac{1}{2},2;0} \leq C\|w_n^i - w^i\|_{C^0}(|w_n^i|_{\frac{1}{2}} + |w^i|_{\frac{1}{2}})$ by [St2] (3.9).

Let $\mathcal{H}_n := \mathcal{H}_\rho(x_n^1, x_n^2)$, $\mathcal{H} := \mathcal{H}_\rho(x^1, x^2)$, $\mathcal{F}_n := \mathcal{F}_\rho(x_n^1, x_n^2) : A_\rho \to N(\hookrightarrow \mathbb{R}^k)$.

Since $\mathcal{H}_n - \mathcal{H}$ is harmonic on $\mathbb{R}^k$ and $\int_{A_\rho} \langle d\mathcal{H}, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega = o(1)$ as $n \to \infty$,

$$\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega = \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1) = \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_\rho(I_n^1, I_n^2))\rangle d\omega + o(1).$$

Now consider $\xi_n^i := -I_n^i \in \mathcal{T}_{x_n^i}$, and set $\mathbf{J}_n^1 := \mathbf{J}_{\mathcal{F}_\rho}(\xi_n^1, 0)$, $\mathbf{J}_n^2 := \mathbf{J}_{\mathcal{F}_\rho}(0, \xi_n^2)$. Then

$$\begin{aligned}
\int_{A_\rho} \langle d\mathcal{F}_n, d\mathcal{H}_\rho(I_n^1, I_n^2)\rangle d\omega &= \int_{A_\rho} \langle d\mathcal{F}_n, d\mathcal{H}_\rho(I_n^1, 0)\rangle d\omega + \langle d\mathcal{F}_n, d\mathcal{H}_\rho(0, I_n^2)\rangle d\omega \\
&= \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\rho} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\rho(I_n^1, 0) + \mathbf{J}_n^1\rangle d\omega \\
&\quad + \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^2\rangle d\omega + \int_{A_\rho} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\rho(0, I_n^2) + \mathbf{J}_n^2\rangle d\omega \\
&\leq g_i(x_n^1, x_n^2, \rho)\|\xi_n^i\|_{\frac{1}{2},2;0} + C(\|\mathcal{F}_n\|_{1,2;0})\|\xi_n^i\|_{C^0} \\
&\leq Cg_i(x_n)\|\xi_n^i\|_{\frac{1}{2},2;0} + C(\|\mathcal{F}_n\|_{1,2;0})\|x_n^i - x^i\|_{C^0},
\end{aligned}$$

where C is independent of $n \geq n_0$, for some n_0. This follows from the observation (Remark 2.1, Remark 2.2 and Lemma 3.2) that $\|x_n^i - x^i\|_{C^0} \to 0$ implies the uniform convergence of $g_i(x_n^1, x_n^2, \rho_{n'}))$ on $\{x_n^i | n \geq n_0\}$ as $\rho_{n'} \to \rho$. Moreover $\|\xi_n^i\|$ are uniformly bounded.

Therefore $\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega \to 0$, and $x_n^i \to x^i$ strongly in $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$.

(case 2) Let $\rho \in (0,1)$, $\|x_n^1 - x^1\|_{C^0} \to 0$, $x_n^2 = \gamma_2 \circ w_n^2 \to \text{const.} = a_2 \in \Gamma_2$ in $L^1(\partial B, \mathbb{R}^k)$.

I) We first claim that $\mathcal{F} := \mathcal{F}_\rho(\gamma^1 \circ w^1, a^2)$ is well defined and conformal. The proof is split into four steps I-a) — I-d) .

I-a) Let $x_n^2 := \gamma_2 \circ w_n^2$, $a_2 := \gamma^2 \circ w^2$ and $\mathcal{F}_{\rho_n} := \mathcal{F}_{\rho_n}(x_n^1, x_n^2)$.

There must exist $\theta_0 \in [0, 2\pi](\cong \partial B)$ such that $\left|\lim_{\theta \to \theta_0 +} w^2(\theta) - \lim_{\theta \to \theta_0 -} w^2(\theta)\right| = 2\pi$. By the Courant-Lebesgue Lemma, for given $\varepsilon > 0$ there exists $r_n \in (\delta, \sqrt{\delta})$ for small $\delta := \delta(\varepsilon) > 0$ such that with $B_{r_n} := B_{r_n}(\theta_0) \subset \mathbb{R}^2$

$$\operatorname{osc}_{A_{\rho_n} \cap \partial B_{r_n}} \mathcal{F}_{\rho_n}(x_n^1, x_n^2) \leq C \, \frac{\mathcal{E}(x_n^1, x_n^2, \rho_n)}{\ln(\delta^{-1})} \leq \frac{C}{\ln(\delta^{-1})} < \varepsilon. \tag{20}$$

For $\varepsilon := \frac{1}{n}$, $C_n^2 := \partial B_{\rho_n} \backslash B_{r_n} \cup (A_{\rho_n} \cap \partial B_{r_n})$, $Y_n^2 := \mathcal{F}_{\rho_n}(C_n^2)$ we see that $\operatorname{dist}(Y_n^2, a_2) \to 0$ as $n \to \infty$, and the energy of $\mathcal{F}_{\rho_n}|_{C_n^2}$ converges to 0.

I-b) Let $\mathcal{H}_{\rho_n} := \mathcal{H}_{\rho_n}(x_n^1, x_n^2)$, $\tilde{\mathcal{H}}_n := \mathcal{H}_{\rho_n}(x^1, a^2)$, $\mathcal{F}_{\rho_n} := \mathcal{F}_{\rho_n}(x_n^1, x_n^2)$. As above, we can say

$$\begin{aligned}\int_{A_{\rho_n}\setminus B_{r_n}} |d(\mathcal{H}_{\rho_n} - \tilde{\mathcal{H}}_n)|^2 d\omega &= \int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d(\mathcal{H}_{\rho_n} - \tilde{\mathcal{H}}_n)\rangle d\omega + o(1) \\ &= \int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathcal{K}_{\rho_n}(I_n^1, \mathcal{F}_{\rho_n}|_{C_n^2} - a^2)\rangle d\omega + o(1),\end{aligned}$$

where $K_{\rho_n}(I_n^1, \mathcal{F}_{\rho_n}|_{C_n^2} - a^2) : A_{\rho_n}\setminus B_{r_n} \to \mathbb{R}^k$ denotes the Euclidean harmonic extension with I_n^1 on ∂B and $\mathcal{F}_{\rho_n}|_{C_n^2} - a^2$ on C_n^2.

Let $\tilde{\mathbf{J}}_n := \mathbf{J}_{\mathcal{F}_{\rho_n}}(\xi_n^1, 0)$ with $\xi_n^1 := -I_n^1$ and $l_n := \tilde{\mathbf{J}}_n|_{C_n^2}$. Since $\|x_n^1 - x^1\|_{C^0} \to 0$, it follows $\|I_n^1\|_{C^0} \to 0$ as $n \to \infty$. We can then estimate further

$$\begin{aligned}&\int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathcal{K}_{\rho_n}(I_n^1, \mathcal{F}_{\rho_n}|_{C_n^2} - a^2)\rangle d\omega \\ &= \int_{A_{\rho_n}\setminus B_{r_n}} -\langle d\mathcal{F}_{\rho_n}, d\tilde{\mathbf{J}}_n\rangle d\omega + \int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathcal{K}_{\rho_n}(I_n^1, -l_n) + d\tilde{\mathbf{J}}_n\rangle d\omega \\ &\qquad + \int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathcal{K}_{\rho_n}(0, l_n + \mathcal{F}_{\rho_n}|_{C_n^2} - a^2)\rangle d\omega \\ &= \int_{A_{\rho_n}\setminus B_{r_n}} -\langle d\mathcal{F}_{\rho_n}, d\tilde{\mathbf{J}}_n\rangle d\omega + \int_{A_{\rho_n}\setminus B_{r_n}} \langle II \circ \mathcal{F}_{\rho_n}(d\mathcal{F}_{\rho_n}, d\mathcal{F}_{\rho_n}), \mathcal{K}_{\rho_n}(I_n^1, -l_n) + \mathbf{J}_n^1\rangle d\omega + o(1) \\ &\qquad \text{(observing that } \int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathcal{K}_{\rho_n}(0, l_n + \mathcal{F}_{\rho_n}|_{C_n^2} - a^2)\rangle d\omega = o(1)) \\ &= \int_{A_{\rho_n}\setminus B_{r_n}} -\langle d\mathcal{F}_{\rho_n}, d\tilde{\mathbf{J}}_n\rangle d\omega + o(1).\end{aligned}$$

Notice that $\int_{A_{\rho_n}\cap B_{r_n}} -\langle d\mathcal{F}_{\rho_n}, d\tilde{\mathbf{J}}_n\rangle d\omega \to 0$ as $n \to \infty$ with $r_n \to 0$, so

$$\begin{aligned}\int_{A_{\rho_n}\setminus B_{r_n}} |d(\mathcal{H}_{\rho_n} - \tilde{\mathcal{H}}_n)|^2 d\omega &= \int_{A_{\rho_n}} -\langle d\mathcal{F}_{\rho_n}, d\tilde{\mathbf{J}}_n\rangle d\omega + o(1) \\ &\le g_1(x_n^1, x_n^2, \rho_n)\|\xi_n^1\|_{\frac{1}{2},2;0} + o(1).\end{aligned}$$

Therefore $\lim_{n\to\infty} \int_{A_{\rho_n}\setminus B_{r_n}} |d(\mathcal{H}_{\rho_n} - \tilde{\mathcal{H}}_n)|^2 d\omega = 0$ and $x_n^1 \to x^1$ strongly in $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$. Moreover, by Lemma 2.2 the N-harmonic map $\mathcal{F}_\rho(x^1, a^2)$ is well defined.

I-c) We shall investigate the behaviour of Jacobi fields.

For large $n \ge n_0$, $\widetilde{\exp}_{x^1}\eta_n^1 = x_n^1$ for some $\eta_n^1 \in \mathcal{T}_{x^1}$, with $\|d\widetilde{\exp}_{x^1,\eta_n^1}\phi^1\| < l_1, \|\phi^1\| < l_1$. Since $x_n^1 \to x^1$ in $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$, $d\widetilde{\exp}_{x^1,\eta_n^1} \to Id$ in $H^{\frac{1}{2},2} \cap C^0$. Defining $(v_n^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}_{\rho_n}) := \mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\eta_n^1}\phi^1, 0)$ we have

$$\int_{A_{\rho_n}} h_{\alpha\beta} \circ \mathcal{F}_{\rho_n} v_{n,i}^\alpha v_{n,i}^\beta d\omega \le C \text{ independent of } n \ge n_0.$$

From the Courant-Lebesgue Lemma and $v_n^\alpha|_{\partial B_{\rho_n}} \equiv 0$,

$$\int_{\partial(B_{\tilde{r_n}} \cap A_{\rho_n})} h_{\alpha\beta} \circ \mathcal{F}_{\rho_n} \partial_\theta v_n^\alpha \partial_\theta v_n^\beta d\theta \leq \frac{C}{|\ln\delta|} \text{ and } \|(v_n^\alpha)\|_{C^0(B_{\tilde{r_n}(\theta_0)} \cap A_{\rho_n})} \leq \frac{C}{|\ln\delta|}$$

for some $\tilde{r_n} \in (\sqrt{\delta}, \sqrt{\sqrt{\delta}})$. Hence, from Lemma 2.6, $E(\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\eta_n^1}\phi^1, 0)|_{B_{\tilde{r_n}}})$ is less than $\frac{C}{|\ln\delta|}$. The same holds for $E(\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\eta_n^1}\phi^1, 0)|_{B_{r_n}})$, since $r_n \leq \tilde{r_n}$. Now choose δ so that $\frac{C}{|\ln\delta|} \leq \varepsilon := \frac{1}{n}$.

I-d) Let $\mathcal{F}_{\rho_n} := \mathcal{F}_{\rho_n}(x_n^1, x_n^2)$. The Hölder inequality gives

$$\begin{aligned}
0 &= \lim_{n\to\infty} g^1(x_n^1, x_n^2, \rho_n) \\
&\geq \lim_{n\to\infty} \Big(-\int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\eta_n^1}\phi^1, 0)\rangle d\omega - \int_{B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\eta_n^1}\phi^1, 0)\rangle d\omega \Big) \\
&= \lim_{n\to\infty} \Big(-\int_{A_{\rho_n}\setminus B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1,\eta_n^1}\phi^1, 0)\rangle d\omega - o(1) \Big) \\
&= -\int_{A_\rho} \langle d\mathcal{F}, d\mathbf{J}_{\mathcal{F}}(\phi^1, 0)\rangle d\omega.
\end{aligned}$$

The computation in Theorem A.1 yields $\mathcal{F} := \mathcal{F}_\rho(x^1, a_2) \in H^{2,2}(A_\rho, N)$. Just as in Proposition 3.1 we have $\langle \frac{d\mathcal{F}}{d\vec{n}}, \frac{\partial\mathcal{F}}{\partial\theta}\rangle_h|_{\partial B} \equiv 0$, and clearly $\langle \frac{d\mathcal{F}}{d\vec{n}}, \frac{\partial\mathcal{F}}{\partial\theta}\rangle_h|_{\partial B_\rho} \equiv 0$.

As a consequence

$$\Phi_{\mathcal{F}}(re^{i\theta}) = r^2\Big|\frac{\partial}{\partial r}\mathcal{F}\Big|_h^2 - \Big|\frac{\partial}{\partial\theta}\mathcal{F}\Big|_h^2 - 2ir\Big\langle \frac{\partial}{\partial r}\mathcal{F}, \frac{\partial}{\partial\theta}\mathcal{F}\Big\rangle_h$$

is real constant.

Going back to the expression for $\frac{\partial}{\partial\rho}\mathcal{E}$ in Lemma 3.1, the holomorphic function $\Phi_{\mathcal{F}}$ must be 0, provided we show $\frac{\partial}{\partial\rho}\mathcal{E}(x^1, a_2, \rho) = 0$. For this one can adapt the argument of [St3]. Thus $\mathcal{F} := \mathcal{F}_\rho(x^1, a_2)$ is conformal.

II) A harmonic, conformal map $\mathcal{F} := \mathcal{F}_\rho(x^1, a_2) \in H^{1,2} \cap C^0(\overline{A_\rho}, N)$ must be constant. To prove this we reproduce Theorem 8.2.3 of [Jo].

Consider the complex upper half-plane $\mathbb{C}^+ = \{\theta + ir | r > 0\}$ and let

$$\mathcal{F}((r+\rho)e^{i\theta}) =: \widetilde{X}(\theta, r), \quad \text{well defined on } \mathbb{R} \times [0, 1-\rho]$$

with $\widetilde{X}(\theta, 0) = \mathcal{F}(\rho e^{i\theta}) \equiv a_2$ and $\frac{\partial^m \widetilde{X}}{\partial\theta^m}|_{\{r=0\}} \equiv 0$ for each m. Choosing an appropriate local coordinate chart in a neighbourhood of a_2, we may assume that $\widetilde{X}(\theta, 0) = 0$. Since $\mathcal{F}$ is conformal and harmonic, $\mathcal{F}|_{A_\rho \cup \partial B_\rho} \in C^\infty([\text{HKW}])$, and by simple computation, $\frac{\partial^m}{\partial\theta^m}\widetilde{X} \equiv \frac{\partial^m}{\partial r^m}\widetilde{X} \equiv 0$ on $\{r = 0\}$, $m \in \mathbb{N}$.

For given $\rho_0 \in (0,1)$, let $\Omega := \{\theta + ir | \theta \in \mathbb{R},\ r \in [0, 1-\rho_0)\}$ and $\Omega^- := \{\theta + ir | \theta \in \mathbb{R},\ -r \in [0, 1-\rho_0)\}$. Extending $\widetilde{X}$ to $\Omega \cup \Omega^- =: \widetilde{\Omega}$ by reflection, we see that $\widetilde{X} \in C^\infty(\widetilde{\Omega}, N)$.

From the harmonicity of $\mathcal{F}$, $|\widetilde{X}_{z\bar{z}}| \le C|\widetilde{X}_z|$ holds. Furthermore $\frac{\partial^m}{\partial\theta^m}\widetilde{X}(0) = \frac{\partial^m}{\partial r^m}\widetilde{X}(0) = 0$ and $\lim_{z=(\theta,r)\to 0}\widetilde{X}(z)|z|^{-m} = 0$ for all $m \in \mathbb{N}$. Hence $\widetilde{X}$ is constant in $\widetilde{\Omega}$ by the Hartman-Wintmer Lemma (see [Jo]). This holds for each $\rho_0 \in (0,1)$, so we get $\mathcal{F} \equiv a_2$ on $\overline{A_\rho}$. But this contradicts the assumption $\operatorname{dist}(\Gamma_1, \Gamma_2) > 0$. Therefore case 2 cannot really occur.

(case3) Suppose that $x_n^i = \gamma_i \circ w_n^i \to$ const.=: $a_i \in \Gamma_i$ in $L^1(\partial B, \mathbb{R}^k)$, $i = 1, 2$. Similarly to case 2, this will lead to a contradiction.

First of all $\Phi_{\mathcal{F}}$ is a real constant for $\mathcal{F} := \mathcal{F}(a^1, a^2)$. Supposing that $\frac{d}{d\rho}E(\mathcal{F}) \neq 0$, we have $\left|\int_0^{2\pi}\int_{\rho+\delta}^{1-t}\left[\left|\frac{\partial}{\partial r}\mathcal{F}_{\rho_n}\right|_h^2 - \frac{1}{r^2}\left|\frac{\partial}{\partial r}\mathcal{F}_{\rho_n}\right|_h^2\right]\frac{1}{1-\rho-\delta}drd\theta\right| = C > 0$ for some fixed $t, \delta > 0$ and large $n \ge n_0$. Let

$$\widetilde{\mathcal{F}}_n^\sigma := \begin{cases} \mathcal{F}_{\rho_n} & \text{on } A_{1-t}, \\ \mathcal{F}_{\rho_n} \circ \tau_{\sigma;1-t}^{\rho+\delta} & \text{on } A_\sigma \backslash A_{1-t}, \\ \mathcal{F}_{\rho_n}(\frac{\rho+\delta}{\sigma}r, \theta) & \text{on } A_{\frac{\sigma\rho_n}{\rho+\sigma}} \backslash A_\sigma, \end{cases}$$

where $\tau_{\sigma;1-t}^{\rho+\delta}$ is a diffeomorphism from $[\sigma, 1-t]$ to $[\rho+\delta, 1-t]$. Then

$$2\frac{d}{d\sigma}E(\widetilde{\mathcal{F}}_n^\sigma)|_{\sigma=\rho+\delta} = \int_0^{2\pi}\int_{\rho+\delta}^{1-t}\left[|\partial_r\mathcal{F}_{\rho_n}|^2 - \frac{1}{r^2}|\partial_\theta\mathcal{F}_{\rho_n}|^2\right]\frac{1-t}{1-t-\rho-\delta}drd\theta.$$

Since $\widetilde{\mathcal{F}}_n^{\rho+\delta} = \mathcal{F}_{\rho_n}$ it follows that

$$\rho_n|g_3(x_n)| = |\rho_n\frac{d}{d\sigma}E(\mathcal{F}_{\rho_n})|_{\sigma=\rho_n}| = |(\rho+\delta)\frac{d}{d\sigma}E(\widetilde{\mathcal{F}}_n^{\rho+\delta})| \ge C > 0,$$

contradicting the assumption $g_3(x_n) \to 0$. Thus, $\mathcal{F}_\rho(a_1, a_2)$ is conformal, and we can use the argument of **(case2)**-II).

(case4): Suppose that $\rho = 0$.

For conformal diffeomorphisms τ_n^i of B, $\mathcal{F}^i(x_n^i) \circ \tau_n^i = \mathcal{F}^i(\widetilde{x_n^i})$ holds with $\widetilde{x_n^i} \in M^{i*}$, $i = 1, 2$. Furthermore $\widetilde{x_n^i}$ has a subquence converging to $x^i \in M^{i*}$ uniformly.

For given $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that for $n \ge n_0$, $(x_n^1, x_n^2, \rho_n) \in \mathcal{N}_\delta(\widetilde{x_n^1}, \widetilde{x_n^2}, 0)$ and $|g(x_n^1, x_n^2, \rho_n) - g(\widetilde{x_n^1}, \widetilde{x_n^2}, 0)| < \varepsilon$. Thus, from the topology of $\overline{\mathcal{M}^*}$ we have then $g(\widetilde{x_n^1}, \widetilde{x_n^2}, 0) \to 0$ when $n \to \infty$.

Once again as in (case 1), some subsequence of $\widetilde{x_n^i}$ strongly converges to $x^i \in M^{i*}$ with $g(x^1, x^2, 0) = 0$. This finishes the proof of the PS condition. □

4.2 Unstable minimal surfaces of annulus type

This section contains three Lemmata, adapted from [St3] to our purposes, in preparation to the main theorems. Before that, we need some explanation for $\overline{\mathcal{M}^*}$ (see section 4.1).

I) The boundary $\partial\mathcal{M}^*$:

(i) For an element $x^i \in M^i$, $(x^i)^{-1}(P_k^i)$ is a closed interval on the unit circle, since x^i is weakly monotone. Let Q_k^i be the first endpoint of $(x^i)^{-1}(P_k^i)$ relative to the positive orientation of the circle for each $i = 1, 2, \ k = 1, 2, 3$. Taking the conformal linear fractional transformation T_{x^i} of the unit disc which maps $(\cos \frac{2\pi(k-1)}{3}, \sin \frac{2\pi(k-1)}{3})$ to Q_k^i and the unit circle onto itself, we have $x^i \circ T_{x^i} \in M^{i*}$. Moreover $T_{x^i \circ T_{x^i}} = Id$, since T_{x^i} is one-to-one.

For $x^i, y^i \in M^i$, we write $x^i \sim y^i$ if $x^i \circ T_{x^i} = y^i \circ T_{y^i}$, clearly an equivalence relation. Now we can quotient M^i in such a way that each class possesses exactly one element from $x^i \in M^{i*}$, denoted by $[x^i] \in M^{i*}$, with $\|[x^i]\| = \|x^i\|$.

(ii) For $x^i \in M^{i*}$ and $\xi^i \in \mathcal{T}_{x^i}, \|\xi^i\| < l_i$ we may calculate $[\widetilde{\exp}]_{[x^i]}\xi^i := [\widetilde{\exp}_{x^i}\xi^i] = [\tilde{x}^i] \in M^{i*}$, where $\widetilde{\exp}_{x^i}\xi^i \circ T_{\widetilde{\exp}_{x^i}\xi^i} = \tilde{x}^i \in M^{i*}$. We will denote this correspondence simply by $\widetilde{\exp}_{x^i}\xi^i = \tilde{x}^i$, which is clearly smooth, since T_{x^i} varies smoothly with $x^i \in M^i$ (cf. above).

Now, for $[x] = ([x^1], [x^2], 0) \in \partial\mathcal{M}^*$ with $x^i \in M^{i*}$, we define $g([x]) := g(x)$, where $x = (x^1, x^2, 0)$. Recall that the Dirichlet integral is invariant under conformal mappings, so for $\xi^1 \in \mathcal{T}_{x^1}$

$$\begin{aligned}
&\mathcal{E}([\widetilde{\exp}]_{[x^1]}t\xi^1, [x^2], 0) \\
&\qquad = E(\mathcal{F}^1([\widetilde{\exp}]_{[x^1]}t\xi^1)) + E(\mathcal{F}^2([x^2])) = E(\mathcal{F}^1([\widetilde{\exp}_{x^1}t\xi^1])) + E(\mathcal{F}^2(x^2)) \\
&\qquad = E(\mathcal{F}^1(\tilde{x}_t^1)) + E(\mathcal{F}^2(x^2)) = E(\mathcal{F}^1(\widetilde{\exp}_{x^1}t\xi^1)) + E(\mathcal{F}^2(x^2)) \\
&\qquad = \mathcal{E}(\widetilde{\exp}_{x^1}t\xi^1, x^2, 0),
\end{aligned}$$

where $\widetilde{\exp}_{x^1}t\xi^1 \circ T_{\widetilde{\exp}_{x^1}t\xi^1} = \tilde{x}_t^1 \in M^{1*}$. The same holds for $\mathcal{E}(x^1, \widetilde{\exp}_{x^2}t\xi^2, 0), \ \ \xi^2 \in \mathcal{T}_{x^2}$. Therefore, $g([x])$ is well defined.

II) The interior $\mathcal{M}^*$:

(i) For $x = (x^1, x^2, \rho) \in \mathcal{M}$ let Q_1^1 be the first endpoint of $(x^1)^{-1}(P_1^1)$ relative to the positive orientation on the circle, and r_{x^1} the positive rotation of A_ρ mapping the point $(1, 0)$ to Q_1^1 of the unit circle. Then $x \circ r_{x^1} := (x^1 \circ r_{x^1}, x^2 \circ r_{x^1}, \rho) \in \mathcal{M}^*$ for each $\rho \in (0, 1)$ and $r_{x^1 \circ r_{x^1}} = Id$.

Since (x^1, x^2, ρ) and $(x^1 \circ r_{x^1}, x^2 \circ r_{x^1}, \rho)$ can be naturally identified, it makes sence to define an equivalence relation $x \sim y$ if $x \circ r_{x^1} = y \circ r_{y^1}$ holds, for $x, y \in \mathcal{M}$. In each equivalence class there is exactly one element from $\mathcal{M}^*$.

(ii) For $[x] = ([x^1, x^2], \rho)$ with $(x^1, x^2, \rho) \in \mathcal{M}^*$ and $\xi^1 \in \mathcal{T}_{x^1}$, we compute $[\widetilde{\exp}]_{[x]}(\xi^1, 0, 0) = ([\widetilde{\exp}_{x^1}\xi^1, x^2], \rho) = ([\widetilde{\exp}_{x^1}\xi^1 \circ r, x^2 \circ r], \rho)$, with $r := r_{\widetilde{\exp}_{x^1}\xi^1}$. Denoting this correspondence simply by $\widetilde{\exp}_x(\xi^1, 0, 0) = (\widetilde{\exp}_{x^1}\xi^1 \circ r, x^2 \circ r, \rho) \in \mathcal{M}^*$, $\widetilde{\exp}$ is clearly smooth.

Let $g([x]) := g(x)$ for $x \in \mathcal{M}^*$ and observe that for $\xi^1 \in \mathcal{T}_{x^1}$

$$\begin{aligned}
\mathcal{E}([\widetilde{\exp}]_{[x]}(t\xi^1, 0, 0)) &= \mathcal{E}([\widetilde{\exp}_{x^1}t\xi^1, x^2], \rho) = E(\mathcal{F}_\rho([\widetilde{\exp}_{x^1}t\xi^1, x^2])) \\
&= E(\mathcal{F}_\rho(\widetilde{\exp}_{x^1}t\xi^1 \circ r_t, x^2 \circ r_t) = E((\mathcal{F}_\rho(\widetilde{\exp}_{x^1}t\xi^1, x^2)) \circ r_t) \\
&= E(\mathcal{F}_\rho(\widetilde{\exp}_{x^1}t\xi^1, x^2)) = \mathcal{E}(\widetilde{\exp}_{x^1}t\xi^1, x^2, \rho),
\end{aligned}$$

where $r_t := r_{\widetilde{\exp}_{x^1} t\xi^1}$ is such that $(\widetilde{\exp}_{x^1} t\xi^1 \circ r_t, x^2 \circ r_t, \rho) \in \mathcal{M}^*$. Moreover, for $\xi^2 \in \mathcal{T}_{x^2}$, $[\widetilde{\exp}]_{[x]}(0, \xi^2, 0) = ([x^1, \widetilde{\exp}_{x^2}\xi^2], \rho)$ with $(x^1, \widetilde{\exp}_{x^2}\xi^2, \rho) \in \mathcal{M}^*$, so we can compute as usual, and $g([x])$ is well defined.

Remark 4.1. *Consider $x \in \partial\mathcal{M}^*$ and $y \in \mathcal{M}^*$ as equivalence classes. Then for $\xi = (\xi^1, \xi^2, 0) \in \mathcal{T}_x\partial\mathcal{M}$ and $\xi_\rho = (\xi^1, \xi^2, \rho) \in \mathcal{T}_x\mathcal{M}$ with $\|\xi^i\|_{\frac{1}{2},2;0} \leq l_i$, we have $\widetilde{\exp}_x\xi \in \partial\mathcal{M}^*$ and $\widetilde{\exp}_y\xi_\rho \in \mathcal{M}^*$. Moreover, $\widetilde{\exp}_x$ and $d\widetilde{\exp}_x$ are continuous.*

And now we proceed with the results.

Lemma 4.1. *For any $\delta > 0$, there exists a uniformly bounded, continuous vector field $e_\delta : M^1 \times M^2 \times [0,1) \to \mathcal{T}_{M^1} \times \mathcal{T}_{M^2} \times \mathbb{R}$, satisfying local Lipschitz continuity on $\mathcal{M}$ and $\partial\mathcal{M}$(separably) with the following properties*

(i) for $\beta \in \mathbb{R}$ there exists $\varepsilon > 0$ such that $y_\delta(x) = \left(\widetilde{\exp}_{x^1} e^1_\delta(x^1), \widetilde{\exp}_{x^2} e^2_\delta(x^2), \rho + e^3_\delta(\rho)\right) \in \mathcal{M}(\rho) := \{x \in \mathcal{M} | x = (x^1, x^2, \rho)\}$ for any $x \in \mathcal{M}(\rho)$ with $\mathcal{E}(x) \leq \beta$ and $0 < \rho < \varepsilon$ (that is, e_δ is parallel to $\partial\mathcal{M}$ near $\partial\mathcal{M}$),

(ii) for any such $\beta, \mathcal{E}, x$ and any pair $T = (\tau^1, \tau^2)$ of conformal transformations of B, $y_\delta(x \circ T) = y_\delta(x) \circ T$, where $\mathcal{F}^i\left((x \circ T)^i\right) = \mathcal{F}^i(x^i) \circ T, \quad i = 1, 2,$

(iii) for any $x \in \overline{\mathcal{M}}$, $\langle d\mathcal{E}(x), e_\delta(x)\rangle_{\mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \times \mathbb{R}} \leq \delta - g(x)$,

(iv) for $x \in \mathcal{M}^*$ and $y \in \partial\mathcal{M}^*$, we have $y_\delta(x) \in \mathcal{M}^*$ and $y_\delta(y) \in \partial\mathcal{M}^*$.

Proof. The proof of the analogous result in [St3] can easily be adapted to our setting, because Remark 4.1 holds.

Lemma 4.2. *For a given locally Lipschitz continuous vector field $f : \overline{\mathcal{M}} \to \mathcal{T}_{M^1} \times \mathcal{T}_{M^2} \times \mathbb{R}$ satisfying Lemma 4.1, there exists a unique flow $\Phi : [0, \infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}$ with*

$$\Phi(0, x) = x, \ \ \frac{\partial}{\partial t}\Phi(t, x) = f\left(\Phi(t, x)\right), \quad x \in \overline{\mathcal{M}^*}.$$

Proof. We use Euler's method. Define $\Phi^{(m)} : [0, \infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}, \ m \geq m_0$, by

$$\begin{aligned} \Phi^{(m)}(0, x) &:= x \\ \Phi^{(m)}(t, x) &:= \widetilde{\exp}_{\Phi^{(m)}(\frac{[mt]}{m}, x)}\Big(\frac{mt - [mt]}{m} f\big(\Phi^{(m)}(\frac{[mt]}{m}, x)\big)\Big), \ \ t > 0 \end{aligned} \tag{21}$$

where $[\tau]$ denotes the largest integer which is smaller than $\tau \in \mathbb{R}$. This is well defined due to the convexity of $\mathcal{T}_{x^i}$, $x^i \in M^i, i = 1, 2$ and by Lemma 4.1 *(iv)*.

Recalling the map $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $x^i = \gamma^i \circ w^i, \ \ x^i \in M^i$ (section 2.2 **III)**), consider

$$W^i := \{w^i \in C^0(\mathbb{R}, \mathbb{R}) : \gamma^i \circ w^i = x^i \ \text{ for some } \ x^i \in M^i\}, \quad W := W^1 \times W^2 \times [0, \infty).$$

Let $\gamma(w) := (\gamma^1 \circ w^1, \gamma^2 \circ w^2, \rho)$ for $(w^1, w^2, \rho) =: w \in W$, $\gamma := (\gamma^1, \gamma^2, Id)$ and $\widetilde{f} := (\widetilde{f}^1, \widetilde{f}^2, f^3)$ with $\widetilde{f}^i(w^i) := (d\gamma^i)^{-1}(f^i(x^i)) \in C^0(\mathbb{R}/2\pi, \mathbb{R})$. Then there exists $\widetilde{\Phi}^{(m)}(t, w) \in W$ with $\Phi^{(m)}(t, x) = \gamma(\widetilde{\Phi}^{(m)}(t, w))$, so we can write (21) as follows:

$$\widetilde{\Phi}^{(m)}(t, w) \quad = \quad \widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w) + \frac{mt - [mt]}{m} \widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w)\big) + 2\pi l, \;\; l \in Z.$$

When $t \in (\frac{k}{m}, \frac{k+1}{m}], k \in \mathbb{Z}$, $\widetilde{\Phi}^{(m)}(t, w) = \widetilde{\Phi}^{(m)}(0, w) + \int_0^t \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{[ms]}{m}, w))ds$.

The compution now proceeds as in the Euclidean case. For any $T > 0$, $G > 0$, there exists $C(T, G)$ with $\|\Phi^{(m)}(\cdot, w)\|_{L^\infty([0,T] \times W_K, \overline{\mathcal{M}})} \leq C(T, G)$, $\;w \in W$ with $\|w\|_W \leq G$.

Let L_1 resp. L_2 be the Lipschitz constants of f in $\{x \in \mathcal{M} \,|\, \|x\| \leq C(T, G)\}$ and $\{x \in \partial\mathcal{M} \,|\, \|x\| \leq C(T, G)\}$, and call $L := \max\{C(\gamma^i)L_1, C(\gamma^i)L_2\}$.

For $\frac{m}{n} < 1$, $\;\|\widetilde{\Phi}^{(m)}(t, w) - \widetilde{\Phi}^{(n)}(t, w)\| \leq tL\frac{2}{m}C(f) + tL\|\widetilde{\Phi}^{(m)}(\cdot, w) - \widetilde{\Phi}^{(n)}(\cdot, w)\|_{L^\infty([0,t],W)}$. Hence, for $m, n \geq m_0$, we have

$$\|\widetilde{\Phi}^{(m)}(\cdot, w) - \widetilde{\Phi}^{(n)}(\cdot, w)\|_{L^\infty([0,t],W)} \leq tL(\frac{2}{m} + \frac{2}{n})C(f) + tL\|\widetilde{\Phi}^{(m)}(\cdot, w) - \widetilde{\Phi}^{(n)}(\cdot, w)\|_{L^\infty([0,t],W)}.$$

By choosing $t \leq \min\{T, \frac{1}{2L}\}$, $\{\widetilde{\Phi}^{(m)}\}$ converges uniformly to some function $\widetilde{\Phi}$ on $[0, t] \times \{w \in W : \|w\| \leq G\}$ as $m \to \infty$. Then $\frac{\partial}{\partial t}\widetilde{\Phi}(t, w) = \widetilde{f}\big(\widetilde{\Phi}(t, w)\big)$.

For $\Phi(t, w) := \gamma \circ \widetilde{\Phi}(t, w) \in \overline{\mathcal{M}}^*$, the uniform boundedness of f yields a flow Φ such that $\frac{\partial}{\partial t}\Phi(t, w) = d\gamma\Big(\widetilde{f}\big(\widetilde{\Phi}^{(m)}(t, w)\big)\Big) = f\big(\Phi(t, w)\big)$ for each $x \in \overline{\mathcal{M}}$. $\Phi(t, w)$ depends continuously on the initial data, and it can be prolonged for $t > 0$. $\square$

The next result is slightly weaker than the corresponding Lemma 4.15 in [St3], but will nevertheless suffice for our aim.

Lemma 4.3. *Let $\mathcal{F}^i(x_0^i)$ be a solution of $\mathcal{P}(\Gamma_i)$ for some $x_0^i \in M^i$, $i = 1, 2$, and suppose that $d := dist(\mathcal{F}^1(x_0^1), \mathcal{F}^2(x_0^2)) > 0$. Then there exist $\varepsilon > 0$, $\rho_0 \in (0, 1)$ and $C > 0$, dependent on $\mathcal{E}(x_0^1, x_0^2, 0)$ such that for $x^i \in M^i$ with $\|x^i - x_0^i\|_{\frac{1}{2},2;0} =: s(x^i) < \varepsilon$,*

$$\mathcal{E}(x^1, x^2, \rho) \geq \mathcal{E}(x^1, x^2, 0) + \frac{Cd^2}{|\ln \rho|}, \quad \text{for all } \rho \in (0, \rho_0).$$

Proof. Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$ $\mathcal{F}^i := \mathcal{F}^i(x^i), i = 1, 2$. Choose σ_1 and δ such that $\sqrt{\rho} < \delta < \sigma_1 < \sqrt{\sqrt{\rho}}$. For $T(re^{i\theta}) := \rho\frac{1}{re^{i\theta}}$ and $\sigma_2 := \frac{\rho}{\delta}$, take $f_{\sigma_1} := \mathcal{F}_\rho|_{A_{\sigma_1}}$ and $g_{\sigma_2} := \mathcal{F}_\rho|_{B_\delta \setminus B_\rho}(T^{-1})$. Then

$$E(\mathcal{F}_\rho) = E(f_{\sigma_1}) + E(\mathcal{F}_\rho|_{B_{\sigma_1} \setminus B_\delta}) + E(g_{\sigma_2}). \tag{22}$$

We will estimate $E(\mathcal{F}_\rho)$ in (I) — (III).

(I) Estimate of $E(f_{\sigma_1})$ and $E(g_{\sigma_2})$.

In order to control $E(f_{\sigma_1})$ we take $a_1 \in N$ with $\min_{a \in N} E(\mathcal{F}_{\sigma_1}(x^1, a)) = E(\mathcal{F}_{\sigma_1}(x^1, a_1))$ and let $\mathcal{F}^1_{\sigma_1} := \mathcal{F}_{\sigma_1}(x^1, a_1)$.

Next, define $\widetilde{\mathcal{F}^1_{\sigma_1}} : B \to N$ as follows: Let $\widetilde{\mathcal{F}^1_{\sigma_1}}|_{B\setminus B_{\frac{1}{2}}}$ be $\mathcal{F}^1_{B\setminus B_{\frac{1}{2}}}$, $\widetilde{\mathcal{F}^1_{\sigma_1}}|_{B_{\frac{1}{2}}\setminus B_{\sigma_1}}$ be harmonic on N with $\mathcal{F}^1|_{\partial B_{\frac{1}{2}}}$ on $\partial B_{\frac{1}{2}}$ and $\mathcal{F}^1(0)$ on ∂B_{σ_1}, and set $\widetilde{\mathcal{F}^1_{\sigma_1}}|_{B_{\sigma_1}} \equiv \mathcal{F}^1(0)$. Thus

$$2E(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1) = \underbrace{\int_{B_{\frac{1}{2}}\setminus B_{\sigma_1}} |\nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)|^2 d\omega}_{=:a} + \underbrace{\int_{B_{\sigma_1}} |\nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)|^2 d\omega}_{=:b}.$$

It is easy to see that $b \le C|\sigma_1|^2$, since $\mathcal{F}^1$ is regular on $B_{\frac{1}{2}}$. Notice $\widetilde{\mathcal{F}^1_{\sigma_1}}|_{B_{\frac{1}{2}}\setminus B_{\sigma_1}} \in H^{2,2}$, since $\widetilde{\mathcal{F}^1_{\sigma_1}}|_{\partial B_{\frac{1}{2}}}$ is regular and constant on ∂B_{σ_1}. Thus,

$$\begin{aligned} a &= \int_{\partial B_{\frac{1}{2}}} \langle \nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)\vec{n}, \widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1\rangle d_0 + \int_{\partial B_{\sigma_1}} \langle \nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)\vec{n}, \widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1\rangle d_0 \\ &\le C\|\mathcal{F}^1(0) - \mathcal{F}^1|_{\partial B_{\sigma_1}}\|_{C^0}\sigma_1 \le C|\sigma_1|^2, \quad \text{with } C = C(E(\mathcal{F}^1(x^1))). \end{aligned}$$

Let $\mathcal{F}^1_{\sigma_1}|_{B_{\sigma_1}} \equiv a_1$, so that $E(\mathcal{F}^1) \le E(\mathcal{F}^1_{\sigma_1}) \le E(\widetilde{\mathcal{F}^1_{\sigma_1}})$. From Lemma 4.4,

$$\begin{aligned} E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1) &\le E(\mathcal{F}^1_{\sigma_1}) - E(\mathcal{F}^1) + o_s(1) \\ &\le E(\widetilde{\mathcal{F}^1_{\sigma_1}}) - E(\mathcal{F}^1) + o_s(1) \le E(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1) + o_s(1) \le C|\sigma_1|^2 + o_s(1), \end{aligned} \tag{23}$$

where $o_s(1) \to 0$ as $\|x^1 - x^1_0\|_{\frac{1}{2},2;0} =: s(x^1) \to 0$.

Since $E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)|_{B_{\sigma_1}} \le C|\sigma_1|^2 + o_s(1)$, we have $E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)|_{A_{\sigma_1}} \le C|\sigma_1|^2 + o_s(1)$.

For $X^1 := f_{\sigma_1} - \mathcal{F}^1_{\sigma_1}$,

$$\left|\int_{A_{\sigma_1}} \nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\nabla X^1 d\omega\right| \le C\sigma_1 \left(\int_{A_{\sigma_1}} \left|\nabla(f_{\sigma_1} - \mathcal{F}^1_{\sigma_1})\right|^2 d\omega\right)^{\frac{1}{2}} \le C\sigma_1.$$

On the other hand,

$$\begin{aligned} |a_1 - \mathcal{F}^1(0)|^2 = \left|\int_{\sigma_1}^1 \partial_r(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1_{\sigma_1})dr\right|^2 &\le (1-\sigma_1)\int_{\sigma_1}^1 \left|\nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1_{\sigma_1})\right|^2 dr \\ &\le \frac{1-\sigma_1}{\sigma_1}(E(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1) + E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)) \le C\sigma_1 + o_s(1). \end{aligned} \tag{24}$$

From the above consideratoins

$$\begin{aligned} \Big|\int_{A_{\sigma_1}} \langle \nabla\mathcal{F}^1_{\sigma_1}, \nabla X^1\rangle d\omega\Big| &\le \Big|\int_{A_{\sigma_1}} \langle \nabla\mathcal{F}^1, \nabla X^1\rangle d\omega\Big| + C\sigma_1 \\ &\le \|\nabla\mathcal{F}^1|_{\partial B_{\sigma_1}}\|\|(-a_1 + \mathcal{F}_\rho|_{\partial B_{\sigma_1}})\|\sigma_1 + C\sigma_1 \le C\sigma_1. \end{aligned}$$

With $C \in \mathbb{R}$ depending on $E(\mathcal{F}^1)$

$$E(f_{\sigma_1}) = E(\mathcal{F}^1_{\sigma_1}) + \int_{A_{\sigma_1}} \langle \nabla \mathcal{F}^1_{\sigma_1}, \nabla X^1 \rangle d\omega + E(X^1) \geq E(\mathcal{F}^1) - C\sigma_1. \tag{25}$$

Similarly $E(g_{\sigma_2}) \geq E(\mathcal{F}^2) - C\sigma_2$, and C depends on $E(\mathcal{F}^2)$.

(II) Estimate of $E(\mathcal{F}_\rho|_{B_{\sigma_1}\setminus B_\delta})$.

From (24), $|a_1 - a_2| \geq \big||\mathcal{F}^1(0) - \mathcal{F}^2(0)| - |a_1 - \mathcal{F}^1(0) + \mathcal{F}^2(0) - a_2|\big| \geq d - o_\rho(1) - o_s(1)$.

Let $\mathcal{H}^a_b(f,g)$ be the harmonic map on $B_a\setminus B_b$ in $\mathbb{R}^k$ with boundary f on ∂B_a and g on ∂B_b. Writing $\sigma_1 =: \sigma$, $\frac{\delta}{\sigma^1} =: \tau$, $\mathcal{F}_\rho|_{\partial B_{\sigma_1}} =: p$, $\mathcal{F}_\rho|_{\partial B_\delta} =: q$, we have

$$\begin{aligned}
&\Big|\int \langle \nabla \mathcal{H}^\sigma_\delta(a_1,a_2), \nabla \mathcal{H}^\sigma_\delta(-a_1+p, -a_2+q)\rangle d\omega\Big| \\
&= \Big|\int \langle \nabla \mathcal{H}^1_\tau(0,-a_1+a_2), \nabla \mathcal{H}^1_\tau(-a_1+p(\cdot\sigma), -a_2+q(\cdot\sigma)\rangle d\omega\Big| \\
&\leq \frac{2\pi}{|\ln\tau|}|-a_1+a_2|\big(|-a_1+p(\cdot\sigma)|+|-a_2+q(\cdot\sigma)|\big) \leq C\frac{(o_\rho(1)+o_s(1))}{|\ln\rho|}.
\end{aligned}$$

Moreover

$$E(\mathcal{H}^\sigma_\delta(a_1,a_2)) \geq E(\mathcal{H}^1_\rho(0,-a_1+a_2)) = E\big((-a_1+a_2)\frac{\ln r}{\ln\rho}\big) \geq \frac{\pi d^2}{|\ln\rho|} - C\frac{(o_\rho(1)+o_s(1))}{|\ln\rho|}.$$

Thus,

$$\begin{aligned}
E(\mathcal{F}_\rho|_{B_\sigma\setminus B_\delta}) &\geq E(\mathcal{H}^\sigma_\delta(p,q)) = E\big(\mathcal{H}^\sigma_\delta(a_1,a_2) + \mathcal{H}^\sigma_\delta(-a_1+p,-a_2+q)\big) \\
&\geq \frac{\pi d^2}{|\ln\rho|} - C\frac{o_\rho(1)+o_s(1)}{|\ln\rho|}
\end{aligned} \tag{26}$$

with C depending only on $E(\mathcal{F}^i)$, $i = 1,2$.

(III) Estimate $E(\mathcal{F}_\rho)$.

From (22), (25), (26) and the choice made for σ_i, $i = 1,2$,

$$\begin{aligned}
&\mathcal{E}(x^1,x^2,\rho) \geq \mathcal{E}(x^1,x^2,0) - C\sigma_i + \frac{\pi d^2}{|\ln\rho|} - C\frac{(o_{\rho(1)}+o_{s(1)})}{|\ln\rho|} \\
&\geq \mathcal{E}(x^1,x^2,0) - C(\sqrt{\rho}+\sqrt{\sqrt{\rho}}) + \frac{\pi d^2}{|\ln\rho|} - C\frac{(o_\rho(1)+o_{s(1)})}{|\ln\rho|} \geq \mathcal{E}(x^1,x^2,0) + C\frac{d^2}{|\ln\rho|},
\end{aligned}$$

for $\rho \leq \rho_0$, some small $\rho_0 \in (0,1)$ and $s(x^i)$. □

Lemma 4.4. *With the same notations as in Lemma 4.3,*

$$E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1) = E(\mathcal{F}^1_{\sigma_1}) - E(\mathcal{F}^1) + o_s(1).$$

Proof. Let $G^1 := \mathcal{F}^1(x_0^1)$. Note $\min_{a\in N} E(\mathcal{F}_{\sigma_1}(x_0^1,a)) = E(\mathcal{F}_{\sigma_1}(x_0^1,a^1))$, and let $G^1_{\sigma_1} := \mathcal{F}_{\sigma_1}(x_0^1,a^1)$, $G^1_{\sigma_1}|_{B_{\sigma_1}} \equiv a^1$.

Since $G^1 \in H^{2,2}$

$$0 = \int_B \langle \nabla G^1, \nabla(G^1_{\sigma_1} - G^1)\rangle d\omega = \int_B \langle II \circ G^1(dG^1, dG^1), G^1_{\sigma_1} - G^1\rangle.$$

Note that $\|\mathcal{F}^1_{\sigma_1} - G^1_{\sigma_1}\|_{C^0} \to 0$ when $\|x_0^1 - x^1\|_{\frac{1}{2},2;0} =: s(x^1) \to 0$ just as in Lemma 3.1 *(B)*. Moreover, $\|G^1 - \mathcal{F}^1\|_{1,2;0} \to 0$ as $s(x^1) \to 0$, so by the Hölder inequality,

$$\left| \int_B \langle II \circ \mathcal{F}^1(d\mathcal{F}^1, d\mathcal{F}^1), \mathcal{F}^1_{\sigma_1} - \mathcal{F}^1\rangle d\omega - \int_B \langle II \circ G^1(dG^1, dG^1), G^1_{\sigma_1} - G^1\rangle d\omega \right| = o_s(1).$$

In this way

$$\int_B \langle \nabla \mathcal{F}^1, \nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\rangle d\omega = \int_B \langle II \circ \mathcal{F}^1(d\mathcal{F}^1, d\mathcal{F}^1), \mathcal{F}^1_{\sigma_1} - \mathcal{F}^1\rangle = o_s(1)$$

and

$$\begin{aligned}
2E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1) &= \int_B \langle \nabla \mathcal{F}^1_{\sigma_1}, \nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\rangle d\omega - \int_B \langle \nabla \mathcal{F}^1, \nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\rangle d\omega \\
&= \int_B \langle \nabla \mathcal{F}^1_{\sigma_1}, \nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\rangle d\omega + o_s(1) \\
&= \int_B |\nabla \mathcal{F}^1_{\sigma_1}|^2 d\omega - \int_B \langle \nabla \mathcal{F}^1, \nabla \mathcal{F}^1_{\sigma_1} - \nabla \mathcal{F}^1\rangle d\omega - \int_B |\nabla \mathcal{F}^1|^2 d\omega + o_s(1) \\
&= \int_B |\nabla \mathcal{F}^1_{\sigma_1}|^2 d\omega - \int_B |\nabla \mathcal{F}^1|^2 d\omega + o_s(1).
\end{aligned}$$

$\square$

We eventually arrive at

Theorem 4.1. *Let $\Gamma_1, \Gamma_2 \subset (N,h)$ satisfy (C1) or (C2) and define*

$$\begin{aligned}
d &= \inf\{E(X)\,|\,X \in \mathcal{S}(\Gamma_1,\Gamma_2)\} \\
d^* &= \inf\{E(X^1) + E(X^2)\,|\,X^i \in \mathcal{S}(\Gamma_i), i = 1,2\}.
\end{aligned}$$

If $d < d^*$, there exists a minimal surface of annulus type bounded by Γ_1 and Γ_2.

Proof. The PS condition (Proposition 4.1) and Proposition 3.1 allow to conclude straight away. For details we refer to [St1]. $\square$

Theorem 4.2. *Let $\mathcal{F}^1$, resp. $\mathcal{F}^2$, be an absolute minimizer of E in $\mathcal{S}(\Gamma_1)$, resp. $\mathcal{S}(\Gamma_2)$, and suppose that dist$(\mathcal{F}^1,\mathcal{F}^2) > 0$. Assume furthermore there is a strict relative minimizer of E in $\mathcal{S}(\Gamma_1,\Gamma_2)$. Then there exists either a solution of $\mathcal{P}(\Gamma_1,\Gamma_2)$ which is not a relative minimizer of E in $\mathcal{S}(\Gamma_1,\Gamma_2)$, i.e. an unstable annulus-type-minimal surface, or a pair of solutions to $\mathcal{P}(\Gamma_1)$, $\mathcal{P}(\Gamma_2)$, one of which does not yield an absolute minimizer of E (in $\mathcal{S}(\Gamma_1)$ or $\mathcal{S}(\Gamma_2)$).*

Proof. Indicate $\mathcal{F}^i := \mathcal{F}^i(x^i)$ for some $x^i \in M^{i*}$, $i = 1, 2$. For some $y \in \mathcal{M}^*$, $\mathcal{F}(y)$ is the strict relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$. Clearly, y is also a strict relative minimizer of $\mathcal{E}$ in $\mathcal{M}^*$. For $x = (x^1, x^2, 0)$, consider

$$P = \{p \in C^0([0,1], \overline{\mathcal{M}}) | p(0) = x, p(1) = y\},$$

and

$$\beta := \inf_{p \in P} \max_{t \in [0,1]} \mathcal{E}(p(t)).$$

The PS condition implies that if $\beta > \max\{\mathcal{E}(x), \mathcal{E}(y)\}$, β is a critical value which possesses a non-relative minimum critical point. Actually $\beta > \mathcal{E}(y)$, since y is a strict relative minimizer. See [St1] chapter II and [Ki1] for details on that.

Supposing that any solution of $\mathcal{P}(\Gamma_i)$ is an absolute minimum of E in $\mathcal{S}(\Gamma_i)$, we have a solution of $\mathcal{P}(\Gamma_1, \Gamma_2)$ which is not a relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$, by the E-minimality of harmonic extensions.

It remains to show that $\beta := \inf_{p \in P} \max_{t \in [0,1]} \mathcal{E}(p(t)) > \mathcal{E}(x)$. We only need to consider $q = (q^1, q^2, \rho) \in p([0, 1])$ for some $p \in P$ such that $\mathcal{E}(q^1, q^2, 0) \leq C$, C a constant dependent on N.

Let ε, ρ_0 be as in Lemma 4.3, and consider the set of q's with $\|q^i - \tilde{x}^i\| \geq \varepsilon$ for any absolute minimizer $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, 0)$ of $\mathcal{E}$ in $\partial\mathcal{M}$. Then there exists $\delta_1 > 0$, dependent on ε, such that $\mathcal{E}(q^1, q^2, 0) \geq \mathcal{E}(x) + \delta_1$ for all but finitely many q's. If not, we would have a minimizing sequence converging to some absolute minimizer $\tilde{x}$ by the PS condition (Proposition 4.1) and Proposition 3.1, contradicting the choice of q.

Moreover, from the uniform convergence of $\mathcal{E}$ on a bounded set of q^i (see Lemma 3.1) when $\rho \to 0$, we can choose δ_2, ρ_1 with $\delta_1 - \delta_2 > 0$, such that for all $\rho \in (0, \rho_1)$, $|\mathcal{E}(q^1, q^2, \rho) - \mathcal{E}(q^1, q^2, 0)| \leq \delta_2$.

Let $\bar{\rho} := \min\{\rho_0, \rho_1\}$. If $\|q^i - \tilde{x}^i\| < \varepsilon$ for some $\tilde{x}$ as above, it is easy to see that $\mathcal{E}(q^1, q^2, \bar{\rho}) \geq \mathcal{E}(x) + \delta_3$ with $\delta_3 > 0$, by Lemma 4.3. If that were not so in fact, then $\mathcal{E}(q^1, q^2, \bar{\rho}) \geq \mathcal{E}(q^1, q^2, 0) - \delta_2 \geq \mathcal{E}(x) + \delta_1 - \delta_2$, by the above choices. This completes the proof. □

Now we specialize the main result to the three-dimensional sphere S^3 and hyperbolic space H^3, to which we can apply condition (C1).

Example 4.1. *Let $\Gamma_1, \Gamma_2 \subset B(p, \pi/2)$ for some $p \in S^3$, in other words Γ_1, Γ_2 lie in a hemisphere. Then the conclusion of the main theorem, under those conditions holds.*

If there is exactly one solution to $\mathcal{P}(\Gamma_i)$, $i = 1, 2$, our main theorem guarantees that the existence of a minimal surface of annulus type whose energy is a strict relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$ ensures the existence of an unstable minimal surface of annulus type. From [LJ], a solution to $\mathcal{P}(\Gamma_i)$ is unique in H^3 if the total curvature of Γ_i is less than 4π. Since $i(p) = \infty$ for all $p \in H^3$ we conclude

Example 4.2. *Let Γ_1, Γ_2 possess total curvature $\leq 4\pi$ in H^3 and $dist(\mathcal{F}^1, \mathcal{F}^2) > 0$. If E has a strict relative minimizer in $\mathcal{S}(\Gamma_1, \Gamma_2)$, then there exists an unstable minimal surface of annulus type in H^3.*

A Regularity of the critical points of $\mathcal{E}$

This appendix is devoted to the proof of the following result, full details of which are found in [Ki2].

Theorem A.1. *Let $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times (0,1)$ with $g_i(x) = 0$, $i = 1,2$. Then $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$ belongs to $H^{2,2}(A_\rho, N)$.*

Noting that $\mathcal{F}_\rho$ is harmonic in $N \overset{\eta}{\hookrightarrow} \mathbb{R}^k$, i.e. $\tau_h(f) = 0$, polar coordinates give

$$\begin{aligned}
|\nabla^2 \mathcal{F}_\rho|^2 &= |\partial_r d\mathcal{F}_\rho|^2 + \frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 \\
&\leq \ C(\varepsilon)|\Delta_{\mathbb{R}^k}\mathcal{F}_\rho|^2 + (2+\varepsilon)\frac{1}{r^2}|\partial_\theta d\mathcal{F}_\rho|^2 + C(\varepsilon)\frac{1}{r^2}\frac{1}{r^2}|\partial_\theta \mathcal{F}_\rho|^2 \\
&\leq \ C(\varepsilon, \eta, A_\rho)|d\mathcal{F}_\rho|^2 + C(\varepsilon, \rho)|\partial_\theta d\mathcal{F}_\rho|^2.
\end{aligned}$$

By a well known result of [GT] it suffices to show that

$$\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega \leq C < \infty, \tag{27}$$

where $\Delta_h d\mathcal{F}_\rho := \frac{d\mathcal{F}_\rho(r,\theta+h) - d\mathcal{F}_\rho(r,\theta)}{h}$, $h \neq 0$ and C is independent of h.

Following [Ho], observe that

Remark A.1. *For $\phi = (\phi^1, \phi^2) \in H^{\frac{1}{2},2} \times H^{\frac{1}{2},2}(\frac{\cdot}{\rho})$ define*

$$\mathbf{A}(\mathcal{F}_\rho)(\phi) := -\int_{A_\rho} \langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho\,, d\mathcal{F}_\rho), X\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, dX\rangle d\omega, \tag{28}$$

where X is any mapping in $H^{1,2}(A_\rho, \mathbb{R}^k)$ with $X|_{\partial A_\rho} = \phi$. Then the expression on the right-hand-side only depends on the boundary of X, since $\mathcal{F}_\rho$ in harmonic in N. □

In particular, taking $\phi^i \in H^{\frac{1}{2},2} \cap C^0(\partial B, (x^i)^* T\Gamma_i), i = 1,2$, we consider $X := \mathbf{J}_{\mathcal{F}_\rho}(\phi^1, \phi^2)$, which is tangent to N along $\mathcal{F}_\rho$. Since $\langle II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho\,, d\mathcal{F}_\rho), \mathbf{J}_\rho(\phi^1, \phi^2)\rangle \equiv 0$,

$$\begin{aligned}
\mathbf{A}(\mathcal{F}_\rho)(\phi) \ &= \ \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(\phi^1, 0)\rangle d\omega + \int_{A_\rho} \langle d\mathcal{F}_\rho, d\mathbf{J}_{\mathcal{F}_\rho}(0, \phi^2)\rangle d\omega \\
&= \ \langle \partial_{x^1}\mathcal{E}, \phi^1\rangle + \langle \partial_{x^2}\mathcal{E}, \phi^2\rangle.
\end{aligned} \tag{29}$$

Hence for a critical point $x = (x^1, x^2, \rho)$ of $\mathcal{E}$, $\mathbf{A}(\mathcal{F}_\rho)(\xi) \geq 0$ for all $\xi = (\xi^1, \xi^2) \in \mathcal{T}_{x^1} \times \mathcal{T}_{x^2}$.

Lemma A.1. *For each $P_0 \in \partial A_\rho$ there exist $C_0, \mu, r_0 > 0$ such that for all $r \in [0, r_0]$*

$$\int_{A_\rho \cap B_r(P_0)} (|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, 0)|^2) d\omega \leq C_0 r^\mu \int_{A_\rho} (|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1}, 0)|^2) d\omega. \tag{30}$$

Proof of Lemma A.1

Let $P_0 \in C_1$ fixed, define $B_r := B_r(P_0)$, and

$$\tilde{w_0^1} := Q^{-1} \int_{(B_{2r}\setminus B_r)\cap \partial B} \tilde{w^1} d_o, \quad w_0^1 := \tilde{w_0^1} + Id : \mathbb{R} \to \mathbb{R},$$

where $\int_{(B_{2r}\setminus B_r)\cap\partial B} d_o := Q$. Then

$$\tilde{\xi}_\phi := -\big[\phi(|e^{i\theta} - P_0|)\big]^2 (w^1 - w_0^1)\frac{\partial}{\partial\theta} \circ \bar{w}^1 \in H^{\frac{1}{2},2} \cap C^0(\partial B, {\bar{w}^1}^* T(\partial B)),$$

where $\bar{w}^1$ is a map from ∂B to itself, and $\phi \in C^\infty$ is a non-increasing function of $|z|$ satisfying $0 \le \phi(z) \le 1$, $\phi \equiv 1$ if $|z| \le 2r$, $\phi \equiv 0$ for $|z| \ge 3r$, plus $|d\phi| \le \frac{C}{r}$ and $|d^2\phi| \le \frac{C}{r^2}$.

Since $(1-\phi^2)w^1 + \phi^2 w_0^1 \in W^1_{\mathbb{R}^k}$, we see that $d\gamma^1(\tilde{\xi}_\phi) \in \mathcal{T}_{x^1}$, and $\mathbf{A}(\mathcal{F}_\rho)(d\gamma^1(\tilde{\xi}_\phi), 0) \ge 0$.

For $x_0^1 := \gamma^1(w_0^1)$,

$$x^1 - x_0^1 \;=\; d\gamma^1(w^1 - w_0^1) - \underbrace{\int_{w_0^1}^{w^1}\int_{s'}^{w^1} d^2\gamma^1(s'')ds''ds'}_{=:\alpha(w^1)},$$

and for small $r > 0$,

$$\begin{aligned} \mathbf{A}(\mathcal{F}_\rho)(\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{C_1}, 0) &= \mathbf{A}(\mathcal{F}_\rho)(\phi^2 d\gamma^1(w^1 - w_0^1), 0) - \mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0) \\ &\le -\mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0), \end{aligned}$$

where $\mathcal{F}_\rho^0(A_\rho) \equiv x_0^1 \in \Gamma_1$.

On the other hand, for small $r > 0, \phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)|_{C_2} \equiv 0$, so we can take $\phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0)$ instead of ϕ in the definition of $\mathbf{A}(\mathcal{F}_\rho)$, to the effect that

$$\begin{aligned} \int_{A_\rho} \langle \phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega \le& \int_{A_\rho} \langle \phi^2(\mathcal{F}_\rho - \mathcal{F}_\rho^0), II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)\rangle d\omega \\ & - \int_{A_\rho} \langle 2\phi d\phi(\mathcal{F}_\rho - \mathcal{F}_\rho^0), d\mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0). \end{aligned} \tag{31}$$

Now define a real valued map of $(r, \theta) \in [\rho, 1] \times \mathbb{R}$ as follows:

$$T^1(w^1)(r,\theta) := H_\rho(\tilde{w}, 0)(r,\theta) + Id(r,\theta) \;\text{ with }\; Id(r,\theta) = \theta,$$

where $H_\rho(\tilde{w}, 0)$ is the harmonic extension to $A_\rho \approx [\rho, 1] \times \mathbb{R}/2\pi$ with $\tilde{w}$ on ∂B and 0 on ∂B_ρ.

In order to estimate $-\mathbf{A}(\mathcal{F}_\rho)(\phi^2\alpha(w^1), 0)$, we consider

$$\widetilde{\star\star} := \phi^2 \int_{w_0^1}^{T^1(w^1)} \int_{s'}^{T^1(w^1)} d^2\gamma^1(s'')ds''ds' \in H^{1,2}(A_\rho, \mathbb{R}^k)$$

with $\widetilde{\star\star}|_{C_1} = \phi^2\alpha(w^1), \widetilde{\star\star}|_{C_2} \equiv 0$, where $w_0^1(r,\theta) = \tilde{w_0^1} + Id(r,\theta) = \tilde{w_0^1} + \theta,\ \ (r,\theta) \in [\rho,1]\times\mathbb{R}$. an easy computation shows that

$$\begin{aligned}
|\widetilde{\star\star}| &\leq C(\gamma^1,x^1)\phi^2|H_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2,\\
|d\widetilde{\star\star}| &\leq C(\gamma^1,x^1)|H_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2\phi|d\phi| + C(\gamma^1,x^1)|dH_\rho(\tilde{w^1},0)||H_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2\phi^2,
\end{aligned}$$

and from (31) Young's inequality implies

$$\begin{aligned}
\int_{A_\rho}\langle\phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega \leq \int_{A_\rho}&|d\mathcal{F}_\rho|^2|\mathcal{F}_\rho - \mathcal{F}_\rho^0|\phi^2 d\omega\\
&+\frac{\varepsilon}{5}\int_{A_\rho}|d\mathcal{F}_\rho|^2\phi^2 d\omega + C(\varepsilon)\int_{A_\rho}|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2|d\phi|^2 d\omega\\
&+C\|H_\rho(\tilde{w^1},0) - \tilde{w_0^1}\|_{L^\infty(B_{3r})}\int_{A_\rho}\big(|d\mathcal{F}_\rho|^2\phi^2 + |H_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2|d\phi|^2\big)d\omega\\
&+C\|H_\rho(\tilde{w^1},0) - \tilde{w_0^1}\|_{L^\infty(B_{3r})}\int_{A_\rho}\big(|dH_\rho(\tilde{w^1},0)|^2 + |d\mathcal{F}_\rho|^2\big)\phi^2 d\omega\\
&+C\int_{A_\rho}|H_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2|d\mathcal{F}_\rho|^2\phi^2 d\omega.
\end{aligned}$$

Thus, for a sufficiently small $r \in (0,r_0)$ dependent on ε, C, and the modulus of continuity of $\mathcal{F}_\rho - \mathcal{F}_\rho^0$ and $H_\rho(\tilde{w^1},0) - \tilde{w_0^1}$, we have an estimate:

$$\begin{aligned}
\int_{A_\rho}\langle\phi^2 d\mathcal{F}_\rho, d\mathcal{F}_\rho\rangle d\omega &\leq \varepsilon\int_{A_\rho}\big(|d\mathcal{F}_\rho|^2 + |dH_\rho(\tilde{w^1},0)|^2\big)\phi^2 d\omega\\
&\quad +C(\varepsilon)\int_{A_\rho}\big(|\mathcal{F}_\rho - \mathcal{F}_\rho^0|^2 + |H_\rho(\tilde{w^1},0) - \tilde{w_0^1}|^2\big)|d\phi|^2 d\omega. \qquad (32)
\end{aligned}$$

This corresponds to (5.6) in [St1] (Proposition 5.1, II). A completely similar computation for $\int_{A_\rho}|dH_\rho(\tilde{w^1},0)|^2\phi^2 d\omega$ and $\int_{A_\rho\cap B_r(P_0)}\big(|d\mathcal{F}_\rho|^2 + |H_\rho(\tilde{w^1},0)|^2\big)d\omega$ eventually yields (30). $\square$

Proof of Theorem A.1

We will show (27) by several steps.

(I) With $\Delta_{-h}\Delta_h\mathcal{F}_\rho|_{\partial B} = \Delta_{-h}\Delta_h\gamma^1\circ e^{iw^1}$ and $\Delta_{-h}\Delta_h\mathcal{F}_\rho|_{\partial B_\rho}(\cdot\rho) = \Delta_{-h}\Delta_h\gamma^2\circ e^{iw^2(\cdot)}$,

$$\begin{aligned}
\int_{A_\rho}|\Delta_h d\mathcal{F}_\rho|^2 d\omega &= -\int_{A_\rho}\langle d\mathcal{F}_\rho, d\Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega\\
&= -\int_{A_\rho}\langle II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho), \Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(\Delta_{-h}\Delta_h\mathcal{F}_\rho|_{\partial A_\rho}).
\end{aligned}$$

Denoting $\gamma^1\circ e^{iw^1}$, $\gamma^2\circ e^{iw^2}$ by $\gamma^1(w^1(\theta))$, $\gamma^2(w^2(\theta))$ and $w^i(\cdot+h)$, $w^i(\cdot-h)$ by w^i_+, w^i_-

respectively, we have:

$$\Delta_{-h}\Delta_h\gamma^i(w^i) = \Delta_{-h}\left[d\gamma^i(w^i)\left(\frac{w^i_+ - w^i_-}{h}\right) + \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\right]$$

$$= \quad d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) \underbrace{-\frac{1}{h}\int_{w^i}^{w^i_-} d^2\gamma^i(s')ds'\cdot\Delta_h w^i_- + \Delta_{-h}\left(\frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\right)}_{=:P^i}$$

Clearly $d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) \in H^{\frac{1}{2},2}\cap C^0(\partial B,(x^i)^*T\Gamma_i)$.

Now define a map $S(P^1,0) : A_\rho \to \mathbb{R}^k$ with boundary $(P^1,0)$ as follows:

$$S(P^1,0) := -\frac{1}{h}\int_{T^1(w^1)(\cdot)}^{T^1(w^1_-)} d^2\gamma^1(s')ds'\cdot H_\rho(\Delta_h w^1_-,0) + \Delta_{-h}\big(\frac{1}{h}\int_{T^1(w^1)}^{T^1(w^1_+)}\int_{T^1(w^1)}^{s'} d^2\gamma^1(s'')ds''ds'\big).$$

Similarly, we have a map $S(0,P^2)(\cdot) : A_\rho \to \mathbb{R}^k$ with 0 on C_1 and P^2 on C_2.

By computation, $\frac{h^2}{2}\Delta_{-h}\Delta_h w^i = \frac{1}{2}(w^i_- + w^i_+) - w^i$, and $\frac{1}{2}(w^i_- + w^i_+) \in W^i_{\mathbb{R}^k}$, noting that $W^i_{\mathbb{R}^k}$ is convex. Thus $\frac{h^2}{2}d\gamma^i(w^i)(\Delta_{-h}\Delta_h w^i) \in \mathcal{T}_{x^i}$ by definition of $\mathcal{T}_{x^i}$,.

As $g^1(x) = g^2(x) = 0$, we have

$$\mathbf{A}(\mathcal{F}_\rho)\big(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^i), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\big) \geq 0, \tag{33}$$

where we have dropped the scaling term $(\frac{\cdot}{\rho})$ in relative to second variation. Now

$$\int_{A_\rho}|\Delta_h d\mathcal{F}_\rho|^2 d\omega = -\int_{A_\rho}\langle II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho),\Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(\Delta_{-h}\Delta_h\mathcal{F}_\rho|_{\partial A_\rho})$$

$$= \quad -\int_{A_\rho}\langle II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho),\Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega$$

$$-\mathbf{A}(\mathcal{F}_\rho)(P^1,P^2) - \mathbf{A}(\mathcal{F}_\rho)\big(d\gamma^1(w^1)(\Delta_{-h}\Delta_h w^1), d\gamma^2(w^2)(\Delta_{-h}\Delta_h w^2)\big)$$

$$\leq \quad -\int_{A_\rho}\langle II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho),\Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega - \mathbf{A}(\mathcal{F}_\rho)(P^1,P^2)$$

$$\leq \quad -\int_{A_\rho}\langle II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho),\Delta_{-h}\Delta_h\mathcal{F}_\rho\rangle d\omega \tag{34}$$

$$+\int_{A_\rho}\langle II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho),S(P^1,0)\rangle d\omega + \int_{A_\rho}\langle II\circ\mathcal{F}_\rho(d\mathcal{F}_\rho,d\mathcal{F}_\rho),S(0,P^2)\rangle d\omega \tag{35}$$

$$-\int_{A_\rho}\langle d\mathcal{F}_\rho,dS(P^1,0)\rangle d\omega - \int_{A_\rho}\langle d\mathcal{F}_\rho,dS(0,P^2)\rangle d\omega. \tag{36}$$

For the estimates of these terms we need some preliminaries.

First, let $s(\tau) := \tau \mathcal{F}_{\rho,+} + (1-\tau)\mathcal{F}_\rho$, $0 \le \tau \le 1$. Then

$$
\begin{aligned}
|\Delta_h II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)| &= |\frac{1}{h}\{II \circ \mathcal{F}_{\rho,+}(\mathcal{F}_{\rho,+}, \mathcal{F}_{\rho,+}) - II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)\}| \\
&= |\frac{1}{h}\{II \circ \mathcal{F}_{\rho,+}(d\mathcal{F}_{\rho,+}, d\mathcal{F}_{\rho,+}) - II \circ \mathcal{F}_\rho(d\mathcal{F}_{\rho,+}, d\mathcal{F}_{\rho,+}) + II \circ \mathcal{F}_\rho(d\mathcal{F}_{\rho,+}, d\mathcal{F}_{\rho,+}) - II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, d\mathcal{F}_\rho)\}| \\
&= |dII(\mathcal{F}_\rho) \cdot \Delta_h \mathcal{F}_\rho(d\mathcal{F}_{\rho,+}, d\mathcal{F}_{\rho,+}) + \frac{1}{h}\int_0^1 \int_0^t d^2 II(s(\tau))|\mathcal{F}_{\rho,+} - \mathcal{F}_\rho|^2 d\tau dt (d\mathcal{F}_{\rho,+}, d\mathcal{F}_{\rho,+}) \\
&\qquad + II \circ \mathcal{F}_\rho(\Delta_h d\mathcal{F}_\rho, d\mathcal{F}_{\rho,+}) + II \circ \mathcal{F}_\rho(d\mathcal{F}_\rho, \Delta_h d\mathcal{F}_\rho)| \\
&\le C(\|\mathcal{F}_\rho\|_{C^0(A_\rho)})[|\Delta_h \mathcal{F}_\rho||d\mathcal{F}_{\rho,+}|^2 + |\Delta_h d\mathcal{F}_\rho|(|d\mathcal{F}_{\rho,+}| + |d\mathcal{F}_\rho|)].
\end{aligned}
$$

Letting

$$
-\frac{1}{h}\int_{T^1(w^1)}^{T^1(w^1_-)} d^2\gamma^1(s')ds' := \star \quad \text{and} \quad \frac{1}{h}\int_{T^1(w^1)}^{T^1(w^1_-)} \int_{T^1(w^1)}^{s'} d^2\gamma^1(s'')ds''ds' := \star\star,
$$

we have

$$
|\star| \le C(\gamma^1)|H_\rho(\Delta_{-h}w^1, 0)|, \quad |\star\star| \le C(\gamma^1)|H_\rho(\Delta_h w^1, 0)|,
$$

and

$$
\begin{aligned}
|d\star| &\le C(\|\gamma^1\|_{C^3})\big(|H_\rho(\Delta_{-h}w^1,0)||dH_\rho(w^1_-,0)| + |dH_\rho(\Delta_{-h}w^1,0)|\big), \\
|d\star\star| &\le C(\|\gamma^1\|_{C^2})|H_\rho(\Delta_h w^1,0)|\big(|dH_\rho(\tilde{w^1}_+,0)| + |dH_\rho(\tilde{w^1},0)|\big).
\end{aligned}
$$

With all that, we can estimate (34), (35), (36) using a $C \in \mathbb{R}$, independent of h. All-in-all then,

$$
\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega = \varepsilon C \int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |dH_\rho(\Delta_h w^1, 0)|^2 d\omega + C(\varepsilon)\Xi, \tag{37}
$$

where Ξ stands for:

$$
\begin{aligned}
&\int_{A_\rho} \big(|dH_\rho(\tilde{w}^1_-,0)|^2 + |dH_\rho(\tilde{w}^1_+,0)|^2 + |dH_\rho(\tilde{w}^1,0)|^2 + |dH_\rho(0,\tilde{w}^2_-)|^2 + |dH_\rho(0,\tilde{w}^2_+)|^2 + dH_\rho(0,\tilde{w}^2)|^2 \\
&\quad +||d\mathcal{F}_\rho|^2\big) \cdot \big(|\Delta_h \mathcal{F}_\rho|^2 + |H_\rho(\Delta_{-h}w^1,0)|^2 + |H_\rho(\Delta_h w^1,0)|^2 + |H_\rho(0,\Delta_{-h}w^2)|^2 + |H_\rho(0,\Delta_h w^2)|^2\big) d\omega.
\end{aligned}
$$

(II) On ∂B we know that $\Delta_h(\gamma^i \circ w^i) = d\gamma^i(w^i)\Delta_h w^i + \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'$, so

$$
\Delta_h w^i = |d\gamma^i(w^i)|^{-2}\big[d\gamma^i(w^i) \cdot \Delta_h \mathcal{F}_\rho - d\gamma^i(w^i) \cdot \frac{1}{h}\int_{w^i}^{w^i_+}\int_{w^i}^{s'} d^2\gamma^i(s'')ds''ds'\big]. \tag{38}
$$

Using $T^i(w^i)$ on the right-hand-side of (38), we get an $H^{1,2}(A_\rho, \mathbb{R}^k)$-extension with boundary $\Delta_h w^i$ on C^1 and 0 on C_2, and by D-minimality of the harmonic extension among the maps with same boundary, it follows that

$$
\begin{aligned}
\int_{A\rho} |dH_\rho(\Delta_h w^1,0)|^2 d\omega &\le C\int_{A_\rho} \big[|dH_\rho(w^1,0)|\big(|\Delta_h \mathcal{F}_\rho| + |\star\star|\big) + |d\Delta_h \mathcal{F}_\rho| + |d\star\star|\big]^2 d\omega \\
&\le C\int_{A_\rho} |d\Delta_h \mathcal{F}_\rho|^2 d\omega + C\Xi, \qquad (39)
\end{aligned}
$$

again by Young's inequality. We van attain a similar estimate for $\int_{A\rho} |dH_\rho(0, \Delta_h w^2)|^2 d\omega$. Using the estimate (37) for $\int_{A_\rho} |d\Delta_h \mathcal{F}_\rho|^2 d\omega$, (39) implies

$$\begin{aligned} &\int_{A_\rho} |d\Delta_h \mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |dH_\rho(\Delta_h w^1, 0)|^2 d\omega + \int_{A_\rho} |dH_\rho(0, \Delta_h w^2)|^2 d\omega \\ &\leq \varepsilon C \int_{A_\rho} |d\Delta_h \mathcal{F}_\rho|^2 d\omega + \varepsilon C \int_{A_\rho} |dH_\rho(\Delta_h w^1, 0)|^2 d\omega + \varepsilon C \int_{A_\rho} |dH_\rho(0, \Delta_h w^2)|^2 d\omega + C(\varepsilon)\Xi\,. \end{aligned}$$

For some small $\varepsilon > 0$ in the above formula we get the inequality:

$$\begin{aligned} &\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |dH_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\ &\qquad \leq C(\varepsilon) \int_{A_\rho} \big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_{\rho+}|^2 + |d\mathcal{F}_{\rho-}|^2 + |dH_\rho(\tilde{w^1}, \tilde{w^2}|)^2 + |dH_\rho(\tilde{w^1_+}, \tilde{w^2_+})|^2 \\ &\qquad\qquad + |dH_\rho(\tilde{w^1_-}, \tilde{w^2_-}|^2\big) \cdot \big(|\Delta_h \mathcal{F}_\rho|^2 + |H(\Delta_{-h} w^1, \Delta_{-h} w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big) d\omega. \end{aligned}$$

Extend now $\mathcal{F}_\rho$ to $\mathbb{R}^2 \backslash B_{\rho^2}$ by conformal reflection

$$\begin{aligned} \mathcal{F}_\rho(z) &= \mathcal{F}_\rho\big(\frac{z}{|z|^2}\big), \text{ if } 1 \leq |z| \\ \mathcal{F}_\rho(z) &= \mathcal{F}_\rho\big(\frac{z}{|z|^2}\rho^2\big), \text{ if } \rho^2 \leq |z| \leq \rho. \end{aligned}$$

Choose $r \in \big(0, \min\{\frac{\rho - \rho^2}{2}, r_0\}\big)$, and $\varphi \in C_0^\infty\big(B_{2r}(0)\big)$ with $\varphi \equiv 1$ on $B_r(0)$.

We may cover A_ρ with balls of radius r in such a way that any $p \in A_\rho$ lies in the intersection of at most k balls, for any r as above (recall $\mathbb{R}^2$ is metrizable). Let B^i denote the balls with centres p_i and set $\varphi_i(p) := \varphi(p - p_i)$. Then

$$\begin{aligned} &\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega + \int_{A_\rho} |dH_\rho(\Delta_h w^1, \Delta_h w^2)|^2 d\omega \\ &\leq C\, \Sigma_i \int_{\mathbb{R}^2 \backslash A_{\rho^2}} \big(|\Delta_h \mathcal{F}_\rho|^2 + |H(\Delta_{-h} w^1, \Delta_{-h} w^2)|^2 + |H(\Delta_h w^1, \Delta_h w^2)|^2\big) \varphi_i^2 \cdot \\ &\quad \underbrace{\big(|d\mathcal{F}_\rho|^2 + |d\mathcal{F}_{\rho+}|^2 + |d\mathcal{F}_{\rho-}|^2 + |dH_\rho(\tilde{w^1}, \tilde{w^2})|^2 + |dH_\rho(\tilde{w^1_+}, \tilde{w^2_+})|^2 + |dH_\rho(\tilde{w^1_-}, \tilde{w^2_-})|^2\big)}_{=:\chi} d\omega. \end{aligned}$$

By substituting $|d\mathcal{F}_{\rho+}|^2$ and $|dH_\rho(\tilde{w^1_+}, \tilde{w^2_+})|^2$ (or $|d\mathcal{F}_{\rho-}|^2$ and $|dH_\rho(\tilde{w^1_-}, \tilde{w^2_-})|^2$) in Lemma A.1, we conclude that χ satisfies the growth condition of Morrey. Now applying Morrey's Lemma (1, Lemma 5.4.1 [Mo]) to χ and $(\Delta_h \mathcal{F}_\rho)\varphi_i$, χ and $H(\Delta_{-h} w^1, \Delta_{-h} w^2)\varphi_i$ or χ and $H(\Delta_h w^1, \Delta_h w^2)\varphi_i$, and adding over the index i for some small $r > 0$, we obtain a constant $C > 0$, independent of $|h| \leq h_0$, such that

$$\int_{A_\rho} |\Delta_h d\mathcal{F}_\rho|^2 d\omega \leq C.$$

□

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